Preface

This textbook is devoted to the intermediate–level course on ordinary differential equations. It treats, as standard topics: existence and uniqueness theory, stability theory and short introduction to functional differential equations. A material of the course is very compressed. The limitation of the time for lecturing did not allow giving deeper suggested topics and more examples. But the content of the textbook reflects main knowledge in ODE, which are studied at many universities. Suggested material is self–contained and sufficient for continuing of self–studying.

The author thanks his students who worked through several preliminary editions of the manuscript as it developed in courses taught at the Suranaree University of Technology (Thailand) since 1997. Their questions, objections, and suggestions have helped to smooth out many rough spots. Among the individuals I gratefully acknowledge Wei-Wei who typed the first version of the course and Nikolay P.Moshkin.

There are many books that have influenced on the present textbook. The main influence was by the book of Pontryagin (1974).

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Chapter 1

First order differential equations

Modern technology requires a deeper knowledge of the behavior of real physical phenomena. Today the main way of studying physical processes and obtaining new knowledge is mathematical modelling: using efficient mathematical modelling allows us to reduce time used in investigation and obtaining new results.

Mathematical models of real world phenomenon are formulated as algebraic, differential or integral equations (or a combination of them). These equations are constructed on the basis of our knowledge about physical phenomena. After the construction of equations the study of their properties is necessary. At this stage the theory of ordinary differential equations plays a significant role.

1.1 Introduction

Definition 1.1. *A differential equation is an equation depending on values of unknown functions, its derivatives and independent variables. Ordinary differential equations are differential equations whose unknown functions only depend on one (independent) variable.*

Ordinary differential equations (ODE) are classified according to order. The order of ODE is defined by the largest order of containing derivatives.

Example 1.1.

$$\begin{aligned} y'x^2 + 3y^2 &= 0 \\ y''y + (y')^2 &= 0 \end{aligned}$$

In this chapter we consider first order ODEs. A general form of a first order ODE is

$$\Phi(x, y, y') = 0.$$

Here the function Φ is a function of the three independent variables: the independent variable x, an unknown function $y = y(x)$ and the derivative $y' = y'(x)$.

Definition 1.2. *(Normal form of first order ODE) If an equation has a special the form*

$$M(x, y) + N(x, y)y' = 0,$$

then it is called a quasilinear equation. If the functions M and N have form

$$M = b(x)y + c(x), \quad N = a(x),$$

then this equation is called a linear equation.

The main goal of studying differential equations is to find their solutions.

Definition 1.3. *A solution of ODE is a function* $f(x),\ x \in J$ *that* $\Phi(x, f(x), f'(x)) = 0$ *for any* x *in the interval* J.

Example 1.2. Let us consider the first-order ODE

$$x + yy' = 0.$$

If $y = y(x)$ is a solution of this equation, then

$$\frac{d}{dx}(x^2 + y^2) = 2(x + yy') = 0.$$

It means that $y = f(x)$ is a solution of the equation if and only if

$$x^2 + f^2 = C,$$

where C is a constant. This formula gives the solution $y = f(x)$ implicitly. The solution is

$$y = +\sqrt{C - x^2} \qquad \text{or} \qquad y = -\sqrt{C - x^2}$$

and it is defined only in the interval $(-\sqrt{C},\ \sqrt{C})$.

Example 1.3. (Fundamental theorem of the calculus)

Let the function $g(x)$ in the equation

$$y'(x) = g(x),\ x \in J$$

be continuous in the interval J. For given numbers $c \in J$ and $a \in J$ there is one and only one solution $y = f(x)$ in the interval J that $f(a) = c$. This solution is given by the integral

$$f(x) = c + \int_a^x g(t)\ dt.$$

Example 1.4. Equations of the type

$$y' = h(y)$$

can be solved by using the previous method.

Let $y = f(x),\ x \in J$ be a solution of this equation. Assume that $f'(a) \neq 0,\ (a \in J)$. From the inverse function theorem one can obtain the inverse function $x = \varphi(y)$. Then for the function $\varphi(y)$ one has

$$1 = \frac{dx}{dy}h(y) \qquad \text{or} \qquad \frac{dx}{dy} = \frac{1}{h(y)}$$

and we can use the previous example.

Example 1.5. The previous examples are particular cases of separable equations. Separable equations can be written in the form

$$M(x) + N(y)y' = 0$$

Separable DEs easy can be solved formally. Really, we can rewrite the equation as

$$M(x)dx + N(y)dy = 0.$$

If the functions $M(x)$ and $N(y)$ are continuous, then there exist $\phi(x) = \int M(x)dx$ and $\psi(y) = \int N(y)dy$ (indefinite integrals). It means that the solution $y = f(x)$ satisfies

$$\phi(x) + \psi(y) = C,$$

where C is constant. If $N(y) = \psi'(y) \neq 0$, then the last equation can be solved with respect to $y = f(x)$. Usually the constructed solution is only local, because the implicit theorem only guarantees a local solution.

1.2 First–order linear equations

Definition 1.4. *The ODE*

$$a(x)y' + b(x)y + c(x) = 0$$

is called a first order linear ODE.

If $c(x) \equiv 0$ then it is called a homogeneous equation, and nonhomogeneous otherwise. If $a(x)$ does not vanish in the interval J, then we can rewrite it in the normal form

$$y' = -p(x)y + q(x).$$

Let us consider the problem how to find a solution of the last equation?

At first we take homogeneous case

$$y' = -p(x)y$$

This is separable equation, therefore

$$\frac{1}{y}dy + p(x)dx = 0.$$

After integrating and exponentiating we obtain

$$y = K \cdot \exp(-\int p(x)dx),$$

where K is constant. By using $P(x) = \int p(x)dx$ (indefinite integral) we can get the same formula by another way

$$\frac{d}{dx}(ye^{P(x)}) = y'e^{P(x)} + ye^{P(x)}P'(x) = e^{P(x)}(y' + p(x)y) = 0.$$

It means that the general solution of the homogeneous equation has the form

$$y = K \ \exp(-P(x)).$$

Theorem 1.1. *All solutions of a linear homogeneous equation are of this from.*

Now let us consider the nonhomogeneous equation. Multiplying the equation on $e^{P(x)}$ we have

$$\frac{d}{dx}(ye^{P(x)}) = q(x)e^{P(x)}$$

or after integrating

$$ye^{P(x)} = c + \int q(x)e^{P(x)}dx$$

Here we used indefinite integrals. If we know that $y(a) = y_0$ (it is called an initial value) we can rewrite it as

$$y = e^{-P(x)}(y_0 + \int_a^x q(t)e^{P(t)}dt), \quad P(x) = \int_a^x p(t)dt.$$

Theorem 1.2. *The general solution of the nonhomogeneous linear first order ODE has this form.*

1.2.1 Linear equations (general survey)

Let X and Y be linear spaces.

Definition 1.5. *$A : X \to Y$ is a linear operator if*

- $A(x_1 + x_2) = Ax_1 + Ax_2, \quad \forall x_1, x_2 \in X$
- $A(\lambda x) = \lambda Ax, \qquad \forall \lambda \in R, \ \ x \in X.$

Example 1.6. A is $m \times m$ matrix.

Example 1.7. The mapping $A : C^1(J) \to C(J)$

$$A\phi(x) = a(x)\phi'(x) + b(x)\phi(x),$$

where $\phi \in C^1(J)$, $a, b \in C(J)$

Let us consider the equation (linear equation)

$$Ax = b \tag{1.1}$$

Here $A : X \to Y$, $b \in Y$ and $x \in X$ is unknown.

Theorem 1.3. *Any solution of the linear equation (1.1) can be represented as*

$$x = x_p + x_0,$$

where x_p is a particular solution of the equation and x_0 is some solution of the homogeneous equation $Ax_0 = 0$.

Proof.

1) Let us show that $x = x_p + x_0$ is a solution of (1.1)

$$Ax = A(x_p + x_0) = Ax_p + Ax_0 = b.$$

2) Let x and x_p be solutions of (1.1). Then $x_0 = x - x_p$ satisfies the linear homogeneous equation $Ax_0 = 0$. Really,

$$Ax_0 = A(x - x_p) = Ax - Ax_p = b - b = 0.$$

□

Remark 1.1. *(Consequence) Any initial value problem of the linear ODE*

$$y' = -p(x)y + q(x), \qquad y(x_0) = y_0$$

with $p(x)$, $q(x) \in C[a, b]$ and $x_0 \in [a, b]$ has one and only one solution defined on the interval $[a, b]$.

Exercise 1.1. *Why can the initial value problem*

$$xy' - 2y = 0, \quad x_0 = 0, \quad y_0 = 0$$

have many solutions?

Remark 1.2. *Method of variation of parameter (a way of constructing solutions of nonhomogeneous linear ODE).*

1.2.2 Quasilinear equations, implicit solutions

The quasilinear equation

$$M(x,y) + N(x,y)y' = 0$$

can be rewritten as

$$M(x,y)dx + N(x,y)dy = 0.$$

If there exists a function $U(x,y)$ such that

$$\frac{\partial U}{\partial x} = M, \qquad \frac{\partial U}{\partial y} = N,$$

then this equation is called an exact differential equation. Why? Because it can be written as

$$\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = 0 \quad \Rightarrow \quad dU = 0$$

Therefore any solution $y = f(x)$ of this equation satisfies

$$U(x, f(x)) = C,$$

where C is constant.

Theorem 1.4. *A quasilinear equation with $N, M \in C^1(D)$ and simply connected domain D is exact DE if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If a quasilinear differential equation is not exact, then it can be done exact by multiplying on a function $\mu(x,y)$ and requiring to be exact with $\hat{M} = \mu M$ and $\hat{N} = \mu N$:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}$$

or

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

Definition 1.6. *A function $\mu(x,y)$ such that a quasilinear DE becomes exact DE is called an integrating factor and the function $\Phi(x,y)$ is called an integral if*

$$\Phi_x = \mu M, \qquad \Phi_y = \mu N.$$

Besides "explicit" solutions $y = f(x)$ and "implicit" solutions $\Phi(x,y) = 0$ we will use "parametric" solutions where the representation of a solution is $x = g(t), \ \ y = h(t)$. In this case $x = g(t), \ y = h(t)$ is a solution if

$$g'(t)M(g(t), h(t)) + N(g(t), h(t))h'(t) = 0.$$

For example $x = A\cos t, \ y = A\sin t$ is the parametric solution of the equation

$$x + yy' = 0.$$

1.2.3 Linear fractional equations

Definition 1.7. *Equations of the form*

$$y' = \frac{cx + dy}{ax + by}, \quad ad \neq bc$$

with the constants a, b, c, d are called linear fractional equations.

An integration of this type of equations can be seen on the sequence of manipulations:

$$y = xv(x) \Rightarrow y' = xv' + v \Rightarrow xv' + v = \frac{c + dv}{a + bv}$$

$$xv' = \frac{c + dv - av - bv^2}{a + bv} \Rightarrow \frac{dx}{x} + \frac{(a + bv)dv}{bv^2 + (a - d)v - c} = 0$$

$$\ln x = g(v) + \ln k \Rightarrow x = k \exp g(v)$$

$$g(v) = -\int \frac{(a + bv)dv}{bv^2 + (a - d)v - c}$$

The same method can be applied to any DE $y' = F(x, y)$, which admits the transformation $\hat{x} = kx, \hat{y} = ky$. The method of solving such equations is the same

$$y = xv(x) \quad \Rightarrow \quad xv' + v = F(x, xv).$$

Because $F(\hat{x}, \hat{y}) = F(x, y)$, then $F(x, xv) = F(1, v)$. Thus,

$$xv' = F(1, v) - v \Rightarrow \frac{dx}{x} = \frac{dv}{F(1, v) - v}.$$

Remark 1.3. *More generally, a type of equations*

$$y' = F(x, y)$$

with the function $F(x, y)$, which satisfies $F(kx, k^n y) = k^{n-1} F(x, y)$, can be solved by changing the unknown function $y = u(x)x^n$:

$$y' = x^n u' + nx^{n-1}u = F(x, x^n u) = x^{n-1}F(1, u)$$

or

$$xu' = -nu + f(u)$$

where $f(u) \equiv F(1, u)$. The simplest method for checking that a equation is of this type, is to check the equality

$$xF_x + nyF_y = (n - 1)F(x, y).$$

1.2.4 Complex differential equations

We have considered only real equations and real solutions, i.e. $y \in R^n$. Sometimes it's easier to find complex solutions of real equations and then to pick out from them real solutions. For such approach we need to introduce notion of complex system of differential equations.

Definition 1.8. *A function* $z : J \to \mathrm{C}$

$$z(t) = \phi(t) + i\psi(t)$$

with $\phi : J \to R,\ \psi : J \to R$ *is called a complex function of the real variable* t *in the interval* $J \subset R$.

The same as for real functions one can define

- $Z^{(l)}(t) = \phi^{(l)}(t) + i\psi^{(l)}(t)$;
- $Z(t) \in C^k(J)$ if $\phi(t) \in C^k(J)$ and $\psi(t) \in C^k(J)$.

Remark 1.4. *(Euler formula). Let* $w = u + iv,\ u \in R,\ v \in R,$ *then*

$$e^w = e^u(\cos v + i \sin v).$$

Example 1.8. According to the previous remark, if $\lambda = \mu + i\gamma$, then

$$\frac{d}{dt}(e^{\lambda t}) = \lambda e^{\lambda t}$$

In fact,

$$Z(t) = e^{\lambda t} = e^{ut}(\cos(\gamma t) + i \sin(\gamma t)) = \phi(t) + i\psi(t)$$

where

$$\phi(t) = e^{\mu t}\cos(\gamma t), \quad \psi(t) = e^{\mu t}\sin(\gamma t).$$

Thus,

$$\begin{aligned} Z'(t) &= (\mu e^{\mu t}\cos(\gamma t) + e^{\mu t}(-\gamma \sin(\gamma t))) + i(\mu e^{\mu t}\sin(\gamma t) + e^{\mu t}(\gamma \cos(\gamma t)) \\ &= e^{\mu t}(\mu e^{i\gamma t} + i\gamma(\cos(\gamma t) + i\sin(\gamma t)) \\ &= e^{\lambda t}(\mu + i\gamma) = \lambda e^{lt}. \end{aligned}$$

Exercise 1.2. *Rewrite the complex equation*

$$Z' = Z^2 + iZ$$

as a system of real equations.

1.3 The initial value problem (Cauchy problem)

We study the normal first-order DE

$$y' = F(x, y)$$

The initial value problem consists of finding the solution (or solutions) $y = f(x),\ x \in J$ of the DE, which satisfies the condition

$$f(x_0) = y_0,\ x_0 \in J.$$

The value x_0 is called an initial point, and the number y_0 is called an initial value. They are given. Another name of this problem is a Cauchy problem.

If $F(x,y) = g(x) \in C(J)$ or $F(x,y) = -p(x)y + q(x)$ with $p(x) \in C(J)$, $q(x) \in C(J)$ we have seen that this problem has one and only one solution and these solutions are continuously dependent on the initial values. A problem with these properties is called a well-posed problem.

Definition 1.9. *The Cauchy problem*

$$y' = F(x,y), \quad y(x_0) = c$$

is said to be well-posed in the domain D, if there is one and only one solution $y = f(x,c)$ in D of the given DE $y' = F(x,y)$ for each given $(x_0, c) \in D$, and if this solution varies continuously with respect to c.

Therefore in order to show that the problem is well-posed one needs to prove the following three theorems:

- a theorem of existence
- a theorem of uniqueness
- a theorem of continuity

We start studying the second and the third properties. At first we show that if $F(x,y) \in C(D)$, then it is not sufficient for the uniqueness.

Example 1.9. Let us consider the set of the functions

$$f(x) = (x-c)^3,$$

where c is a parameter. These functions satisfy the equation

$$f'(x) = 3(x-c)^2 = 3f^{2/3}(x).$$

It means that the set $f(x)$ is a set of solutions of the equation $y' = 3y^{2/3}$. Note that the functions ($\alpha < \beta$)

$$y = \begin{cases} (x-\alpha)^3 & x < \alpha \\ 0 & \alpha \le x \le \beta \\ (x-\beta)^3 & x > \beta \end{cases}$$

are also solutions of this equation.

Exercise 1.3. *What the class C^k do these functions belong to?*

If one takes the Cauchy problem

$$y(0) = 0,$$

then this problem has two-parameter set of solutions. Thus, this Cauchy problem has no uniqueness.

1.4 Uniqueness and continuity

We have seen that if $F(x,y) \in C(D)$, then it is not enough for uniqueness of the Cauchy problem with $(x_0, y_0) \in D$. But if $F(x,y) \in C^1(D)$, then one can prove uniqueness and continuity.

Definition 1.10. *A function $F(x,y)$ satisfies a one-sided Lipschitz condition in a domain D if there exists a finite constant L that $y_2 > y_1$ implies*

$$F(x,y_2) - F(x,y_1) \le L(y_2 - y_1)$$

for any $(x, y_2) \in D$ *and* $(x, y_1) \in D$.

Definition 1.11. *(Lipschitz condition). A function* $F(x, y)$ *satisfies a Lipschitz condition in a domain* D *if there exists a finite constant* L *that*

$$|F(x, y) - F(x, z)| \leq L\,|y - z|\,, \qquad \forall (x, y) \in D,\ \ \forall (x, z) \in D.$$

Exercise 1.4. *Show that if* $F(x, y)$ *satisfies a Lipschitz condition, then it satisfies a one-sided Lipschitz condition.*

Example 1.10. Let $F(x, y) = 3y^{2/3}$.

- If $D_1 = \{(x, y)|\ y \geq \varepsilon > 0\}$, then there exists L that $F(x, y)$ satisfies a Lipschitz condition in D_1 ($L = 2\varepsilon^{-1/3}$).
- If $D_2 = \{(x, y)|\, y > 0\}$, then $F(x, y)$ does not satisfy a Lipschitz condition in D_2 (one cannot find the constant L).

In checking that a function satisfies a Lipschitz condition one must find a constant L. The next lemma is about how to find it.

Lemma 1.1. *Let* $F(x, y)$ *be continuously differentiable in a bounded closed convex domain* D ($F \in C^1(D)$). *Then it satisfies a Lipschitz condition there with* $L = \sup_D \left|\frac{\partial F}{\partial y}\right|$.

Proof.
The domain D, being convex, contains the entire vertical segment joining (x, y) with (x, z). Therefore there exists some η between y and z that

$$|F(x, y) - F(x, z)| = |y - z| \left|\frac{\partial F}{\partial y}(x, \eta)\right|.$$

This implies $L = \sup_D \left|\frac{\partial F}{\partial y}\right|$. □

Example 1.11.
a) If $F(x, y) = g(x)$, then $\frac{\partial F}{\partial y} = 0$ and L can be chosen as $L = 0$.
b) If $F(x, y) = -p(x)y + q(x),\ x \in J$ (J is closed, $p \in C(J),\ q \in C(J)$), then $\frac{\partial F}{\partial y} = -p$ and L can be chosen as $L = \max_{x \in J} |p(x)|$.

Remark 1.5. *For satisfaction a Lipschitz condition we needed only continuous differentiability with respect to* y.

Lemma 1.2. *Let* $\sigma(x)$ *be a differentiable function satisfying the differential inequality:*

$$\sigma'(x) \leq K\sigma(x),\ \ x \in [a, b]$$

where K *is a constant. Then*

$$\sigma(x) \leq \sigma(a)e^{K(x-a)}, \quad \forall x \in [a, b].$$

Proof.
After multiplying the inequality $\sigma'(x) - K\sigma(x) \leq 0$ by e^{-Kx} we get

$$0 \geq e^{-Kx}(\sigma'(x) - K\sigma(x)) = \frac{d}{dx}(\sigma(x)e^{-Kx}).$$

It means that the function $\sigma(x)e^{-Kx}$ is nonincreasing on the interval $[a,b]$. Therefore $\sigma(x)e^{-Kx} \leq \sigma(a)e^{-Ka}$. □

Lemma 1.3. *If a function $F(x,y)$ is a one-sided Lipschitz condition function, then for any two solutions $y = g(x)$ and $y = f(x)$ of the equation $y' = F(x,y)$ there is*

$$[g(x) - f(x)][g'(x) - f'(x)] \leq L[g(x) - f(x)]^2.$$

Proof.
There is

$$\begin{aligned} [g(x) - f(x)][g'(x) - f'(x)] &= [f(x) - g(x)][f'(x) - g'(x)] = \\ &= [g(x) - f(x)][F(x, g(x)) - F(x, f(x))]. \end{aligned}$$

If $g(x) > f(x)$, then $[g(x) - f(x)][g'(x) - f'(x)] \leq L[g(x) - f(x)]^2$. □

Exercise 1.5. *How to be in the case $f(x) > g(x)$?*

Theorem 1.5. *Let $f(x)$ and $g(x)$ be any two solutions of the ODE $y' = F(x,y)$ in a domain D and $F(x,y)$ satisfies a one-sided Lipschitz condition with a constant L. Then*

$$|f(x) - g(x)| \leq |f(a) - g(a)|e^{L(x-a)}, \quad \forall x \geq a.$$

Proof.
Let us consider the function $\sigma(x) = [g(x) - f(x)]^2$. By the previous lemmas and because of $\sigma'(x) \leq 2L\sigma(x)$ we have

$$\sigma(x) \leq \exp(2L(x-a))\sigma(a), \quad \forall x > a.$$

It proves the theorem. □

Theorem 1.6. *If $F(x,y)$ satisfies a Lipschitz condition in a domain D, $f(x)$ and $g(x)$ are solutions of the ODE $y' = F(x,y)$. Then*

$$|f(x) - g(x)| \leq \exp(L|x-a|)|f(a) - g(a)|.$$

Proof.
If $F(x,y)$ satisfies a Lipschitz condition, then $F(x,y)$ satisfies a one-sided Lipschitz condition. The proof of the theorem for $x \geq a$ follows from the previous theorem.

If $x \leq a$, then after substitution $t = -x$ one can also apply the previous theorem. In fact, for t there is the same property as before with the solutions $u(t) = f(-t)$ and $v(t) = g(-t)$, $t \geq \hat{a} = -a$ of the equation

$$\frac{dw}{dt} = -F(-t, w).$$

Then

$$|u(t) - v(t)| \leq e^{L(t-\hat{a})}|u(\hat{a}) - v(\hat{a})|, \ t \geq \hat{a}$$

or (for $x \leq a$)

$$|f(x) - g(x)| \leq e^{L(a-x)}|f(a) - g(a)| = e^{L|x-a|}|f(a) - g(a)|.$$

□

Corollary 1.1. *(uniqueness). If $F(x,y)$ satisfies a Lipschitz condition in domain a D, $f(x)$ and $g(x)$ are solutions in the interval $[\alpha, \beta]$ of the Cauchy problem*

$$y' = F(x,y), \quad y(x_0) = y_0, \quad \forall (x_0, y_0) \in D,$$

then $f(x) = g(x)$, $\forall x \in [\alpha, \beta]$.

1.5 Comparison theorems

In this section the differential inequalities

$$f'(x) \leq F(x, f(x))$$

are studied.

Theorem 1.7. *Let $F(x, y)$ satisfy a Lipschitz condition with a constant L for $x \geq a$. If a function $f(x)$ satisfies the differential inequality*

$$f'(x) \leq F(x, f(x)), \quad \forall x \geq a$$

and $g(x)$ is a solution of ODE $g'(x) = F(x, g(x))$, which satisfies the condition $g(a) = f(a)$, then

$$f(x) \leq g(x), \ \forall x \geq a.$$

Proof.
Assume that there is $x_1 \geq a$ such that $f(x_1) > g(x_1)$. We define $x_0 \in [a, x_1]$ that x_0 is the largest x with the property $f(x) \leq g(x)$. We can show that $f(x_0) = g(x_0)$ (prove this as exercise).
Let $\sigma(x) = f(x) - g(x) \geq 0$, $x \in [x_0, x_1]$, then $\sigma(x_1) = 0$. Really, since

$$\begin{aligned} \sigma'(x) &= f'(x) - g'(x) = f'(x) - F(x, g(x)) \\ &\leq F(x, f(x)) - F(x, g(x)) \leq L(f(x) - g(x)) = L\sigma(x), \end{aligned}$$

then

$$\sigma(x) \leq e^{L(x-x_0)}\sigma(x_0), \qquad \forall x > x_0.$$

Thus, $\sigma(x) = 0$, $\forall x > x_0$ and hence, $\sigma(x_1) = 0$. We obtained the contradiction to the hypothesis that $f(x_1) > g(x_1)$. □

Theorem 1.8. *Let $g(x)$ and $f(x)$ be solutions of the ODE's*

$$g'(x) = G(x, g(x)), \quad f'(x) = F(x, f(x)),$$

where $F(x, y) \leq G(x, y)$ in the strip $a \leq x \leq b$ and $F(x, y)$ or $G(x, y)$ satisfies a Lipschitz condition. If $f(a) = g(a)$, then

$$f(x) \leq g(x), \quad \forall x \in [a, b].$$

Proof.
a) Let $G(x, y)$ satisfy a Lipschitz condition. Since

$$f'(x) = F(x, f(x)) \leq G(x, f(x)),$$

then from the previous theorem one has

$$f(x) \leq g(x), \qquad \forall x \geq a.$$

b) Let $F(x, y)$ satisfy a Lipschitz condition. If we take the functions $u = -f(x)$, $v = -g(x)$ and $H(x, u) = -F(x, -u)$, then

$$u' = H(x, u), \quad v' = -G(x, -v) \leq -F(x, -v) = H(x, v)$$

Because $H(x, v)$ satisfies a Lipschitz condition, then

$$v(x) \le u(x), \qquad \forall x \ge a$$

or

$$g(x) \ge f(x), \qquad \forall x \ge a.$$

□

Corollary 1.2. *Let $g(x)$ and $f(x)$ be solutions of the ODE's*

$$g'(x) = G(x, g(x)), \;\; f'(x) = F(x, f(x)),$$

where $F(x, y) \le G(x, y)$ in the strip $a \le x \le b$ and $G(x, y)$ satisfies a Lipschitz condition. If $f(a) = g(a)$, then for any $x_1 > a$, either $f(x_1) < g(x_1)$ or $f(x) \equiv g(x)$, $\forall x \in [a, x_1]$.

Proof.

From the theorem we have that the function

$$\sigma_1(x) = g(x) - f(x) \ge 0, \qquad \forall x \in [a, x_1].$$

Therefore for the point $x_1 \in (a, b]$ there is two possibilities: either $f(x_1) < g(x_1)$ or $f(x_1) = g(x_1)$. In the first case the theorem is proved.

Assume that $f(x_1) = g(x_1)$. In this case one has to prove that $f(x) = g(x)$, $\forall x \in [a, x_1]$. Let $x_0 \in [a, x_1]$ be a point in which $g(x_0) - f(x_0) > 0$. We show that then $g(x) > f(x)$, $\forall x \in [x_0, x_1]$, that contradicts to the assumption $f(x_1) = g(x_1)$. In fact, one has

$$\begin{aligned} \sigma_1'(x) &= g'(x) - f'(x) = G(x, g(x)) - F(x, f(x)) \\ &\ge G(x, g(x)) - G(x, f(x)) \ge -L(g(x) - f(x)) = -L\sigma_1(x). \end{aligned}$$

Thus,

$$(e^{Lx}\sigma_1(x))' = e^{Lx}(\sigma_1'(x) + L\sigma_1(x)) \ge 0.$$

This means that the function $e^{Lx}\sigma_1(x)$ is a nondecreasing function in $[x_0, x_1]$. Consequently, we have

$$\sigma_1(x) \ge \sigma_1(x_0)e^{-L(x-x_0)} > 0, \qquad \forall x > x_0.$$

□

Corollary 1.3. *Let $F(x, y)$ (or $G(x, y)$) satisfy a Lipschitz condition and $F(x, y) \le G(x, y)$ in the strip $a \le x \le b$. If $f(a) < g(a)$, then*

$$f(x) < g(x), \qquad \forall x \in [a, b].$$

Proof.

Assume that $x_1 > a$ is the first x where $f(x) \ge g(x)$. At this point $f(x_1) = g(x_1)$. The functions

$$\phi(t) = f(-t), \quad \psi(t) = g(-t)$$

are solutions of the equations

$$\phi' = -F(-t, \phi), \qquad \psi' = -G(-t, \psi).$$

Since

$$-F(-t, y) \ge -G(-t, y)$$

and $\phi(-x_1) = \psi(-x_1)$, then from the theorem we obtain $\phi(t) \geq \psi(t), \quad t > -x_1$ and, therefore $\phi(-a) \geq \psi(-a)$ or $f(a) \geq g(a)$. It contradicts to the condition $f(a) < g(a)$. □

Exercise 1.6. *Let $f(u)$ be continuous and*

$$a + bf(u) \neq 0, \quad \forall u \in [p, q],$$

where a, b, c are constants. Show that ODE

$$y' = f(ax + by + c),$$

has a solution passing through every point of the strip

$$p < ax + by + c < q.$$

Exercise 1.7. *Let F, G, f, g be as in the last theorem, and $F(x, y) < G(x, y)$. Show that*

$$f(x) < g(x), \quad \forall x > a$$

without assuming that F or G satisfies a Lipschitz condition.

Chapter 2

Existence and uniqueness

2.1 Additional Knowledge (from Real Analysis)

2.1.1 Normed spaces

Let X be a linear space.

Definition 2.1. *Pair $(X, \| \cdot \|)$ is called a normed space if the function $\| \cdot \|: X \to R_+$ has the properties:*

- 1) $\| x \| = 0 \Leftrightarrow x = 0$,
- 2) $\| \lambda x \| = |\lambda| \, \| x \| \quad \forall x \in X, \quad \lambda \in R$ (a homogeneity of the norm),
- 3) $\| x + y \| \leq \| x \| + \| y \| \quad \forall x, y \in X$ (a triangle inequality).

The function $\| \cdot \|$ is called a norm ($\| x \|$ – norm of the vector x).

Exercise 2.1. *Prove that $| \; \| x \| - \| y \| \; | \leq \| x - y \|$, $\forall x, y \in X$.*

Example 2.1. $(R^n, \| \cdot \|_2)$ is a normed space, where $\| x \|_2 = \sqrt{\Sigma_{i=1}^n x_i^2}$ for any vector $x = (x_1, x_2, \cdots, x_n) \in R^n$

Exercise 2.2. *Prove the Schwartz inequality:*

$$|(x, y)| \leq \| x \|_2 \| y \|_2$$

(Hint: Use that $z^2 \geq 0$ of the vector $z = ax - by$ with $a = \| x \|_2$ and $b = \| y \|_2$).

Example 2.2. $(C\,[a, b]\,, \; \| \cdot \|)$ with $\| f(x) \| = \max_{x \in [a,b]} |f(x)| \;\; \forall f \in C([a, b])$.

Exercise 2.3. *Prove that in the previous example $\| \cdot \|$ is a norm.*

2.1.2 Open and closed sets.

Definition 2.2. *A set $U \subset X$ is called an open set in X if*

$$\forall u \in U, \; \exists \varepsilon > 0 \textit{ that } \forall x \in X, \; \| x - u \| < \varepsilon \Rightarrow x \in U.$$

Definition 2.3. *A set $M \subset X$ is called a closed set if there is an open set U that $U \cup M = X$ $(M = X \backslash U)$.*

Example 2.3. The set $U = \{x \in X| \parallel x \parallel < 1\}$ is open in X.
In fact, let $\parallel u \parallel < 1,\ u \in U$ and $x \in X$ that $\parallel x - u \parallel < \varepsilon = 1- \parallel u \parallel$, then

$$\parallel x \parallel = \parallel x - u + u \parallel \leq \parallel x - u \parallel + \parallel u \parallel < 1- \parallel u \parallel + \parallel u \parallel = 1$$

This means that $x \in U$.

Exercise 2.4. *Prove that the set* $M = \{x \in X| \parallel x \parallel \leq 1\}$ *is closed.*

Exercise 2.5. *Prove that if a set* M *is a closed set and* $\{x_n\} \to x_*$, $x_n \in M$, *then* $x_* \in M$.

2.1.3 The Cauchy sequence

Definition 2.4. *A sequence* $\{x_n\} \subset X$ *is called a Cauchy sequence if*

$$\forall \varepsilon > 0,\ \exists N,\ \textit{that}\ \ \forall n, m > N \ \Rightarrow\ \parallel x_n - x_m \parallel < \varepsilon$$

Definition 2.5. *A normed space* $(X, \| \cdot \|)$ *is called a complete normed space if every Cauchy sequence converges (to some element of the space* X*). A complete normed space is called a Banach space.*

Example 2.4. $(C[a,b], \| \cdot \|)$ is a complete space, therefore it is a Banach space.

Exercise 2.6. *Prove that* $(C[a,b], \| \cdot \|)$ *is a Banach space.*

2.1.4 Contraction principle

Definition 2.6. *Let* M *be a closed set in* X. *An operator* $T : M \to M$ *is called a contraction on* M *if there exists a constant* q *that* $0 < q < 1$ *and*

$$\parallel Tx - Ty \parallel \leq q \parallel x - y \parallel,\ \forall x, y \in M.$$

Definition 2.7. *An element* x_* *is called a fixed point of an operator* T *if*

$$x_* = Tx_*.$$

Exercise 2.7. *Prove that* $T(x) = \arctan(x)$ *is not a contraction (although* $|\arctan(x) - \arctan(y)| < |x - y|,\ \forall x, y \in R^1$*).*

Let us consider a sequence $\{x_n\}$ with

$$x_0, x_1 = Tx_0, x_2 = T^2 x_0, \ldots$$

or $x_{n+1} = Tx_n$. The vector $x_0 \in V$ is called an initial element.

Theorem 2.1. *Let* $(X, \| \cdot \|)$ *be a Banach space,* M *be a closed set in* X, *and* $T : M \to M$ *be a contraction. Then there exists one and only one fixed point* $x_* = Tx_*$ *and* $x_* \in M$.

Proof.

Assume that $x, y \in M$. Since $\parallel Tx - Ty \parallel \leq q \parallel x - y \parallel$, then $\forall n$ one has

$$\parallel T^n x - T^n y \parallel \leq q^n \parallel x - y \parallel,\ \ \forall x, y \in M.$$

Let us consider a sequence $\{x_n\}$, where $x_n = T^n x_0$ (or $x_{n+1} = Tx_n,\ \forall n$). One can show that the sequence $\{x_n\}$ is a Cauchy sequence. Let n and p be natural numbers and $m = n + p$. Then

$$\begin{aligned}
\| & \ x_n - x_m \| = \| x_n - x_{n+p} \| = \| x_{n+p} - x_{n+p-1} + x_{n+p-1} - \dots x_{n+1} - x_n \| \leq \\
\leq & \ \| x_{n+p} - x_{n+p-1} \| + \| x_{n+p-1} - x_{n+p-2} \| + \dots + \| x_{n+1} - x_n \| \leq \\
\leq & \ q^p \| x_n - x_{n-1} \| + q^{p-1} \| x_n - x_{n-1} \| + \dots + q \| x_n - x_{n-1} \| \leq \\
= & \ q(q^{p-1} + q^{p-2} + \dots + q + 1) \| x_n - x_{n-1} \| \leq \\
\leq & \ q(1 + q + \dots + q^{p+1} + \dots) \| x_n - x_{n-1} \| = \\
= & \ \frac{q}{1-q} \| x_n - x_{n-1} \| \leq \frac{q}{1-q} (q^{n-1} \| x_1 - x_0 \|) = \frac{q^n}{1-q} \| x_1 - x_0 \|
\end{aligned}$$

This means that $\forall \varepsilon > 0,\ \exists N \ \in \mathbf{N},\ \forall n > N$ that $\frac{q^n}{1-q} \| x_1 - x_0 \| < \varepsilon$ and therefore

$$\forall \varepsilon > 0,\ \exists N \in \mathbf{N} \ \text{ that } \forall n, m \geq N \implies \| x_n - x_m \| < \varepsilon.$$

Thus the sequence $\{x_n\}$ is a Cauchy sequence. Because X is a Banach space, then $\{x_n\} \to x_* \in X$. By virtue of closeness of M and of $x_n \in M$, we obtain $x_* \in M$.

Also one needs to prove that x_* is a unique fixed point. There is the sequence of inequalities

$$\begin{aligned}
\| & \ x_* - Tx_* \| \leq \| x_* - x_n + x_n - Tx_* \| \leq \| x_* - x_n \| + \| Tx_{n-1} - Tx_* \| \\
\leq & \ \| x_* - x_n \| + q \| x_* - x_{n-1} \| \ \to \ 0.
\end{aligned}$$

Therefore $\| x_* - Tx_* \| = 0$ or $x_* = Tx_*$. Assume that there is another point $y \in M$ that $y = Ty$. Then

$$\| x - y \| = \| Tx - Ty \| \leq q \| x - y \|$$

or

$$0 \leq (1 - q) \| x - y \| \leq 0.$$

This means that $\| x - y \| = 0$ or $x = y$. □

Remark 2.1. *In the process of proving the theorem we obtained the inequalities*

$$\| T^{n+p} x - T^n x \| \leq q^n \frac{1 - q^p}{1 - q} \| x - Tx \|$$

$$\| x_* - T^n x \| \leq \frac{q^n}{1 - q} \| x - Tx \|$$

that are valid for any vector $x \in M$.

2.1.5 Continuity of solutions with respect to a parameter

The operator equation

$$x = T_\lambda x$$

with an operator $T_\lambda : M \to M$, depends on the parameter $\lambda \in V$. Here M is a subset of X. For all $\lambda \in V$ the operators T_λ satisfy a contraction property with $0 < q < 1$:

$$\| T_\lambda x - T_\lambda y \| \leq q \| x - y \| .$$

Note that q does not depend on λ.

By virtue of the previous theorem $\forall \lambda \in V,\ \exists x_\lambda = x(\lambda)$ that

$$x_\lambda = T_\lambda x_\lambda$$

We say that solution $x_\lambda = x(\lambda)$ depends on a parameter.

Definition 2.8. *A solution $x(\lambda)$ is continuously dependent on λ at the point λ_0 if*

$$\| x(\lambda) - x(\lambda_0) \| \to 0 \qquad as \quad \lambda \to \lambda_0.$$

If $x(\lambda)$ is continuous at every point $\lambda \in V$, then it is said that $x(\lambda)$ is continuous on V.

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a Banach space and M be a closed set in X. If for each parameter $\lambda \in V$ the operator $T_\lambda : M \to M$ and*

1) for each $\lambda \in V$ the operator $T_\lambda : M \to M$ is a contraction with a constant q:

$$\| T_\lambda x - T_\lambda y \| \le q \| x - y \|, \quad \forall x, y \in M, \quad \lambda \in V,$$

2) $x(\lambda_0) = T_{\lambda_0} x(\lambda_0)$ for $\lambda_0 \in V$,
3) $T_\lambda x(\lambda_0) \to T_{\lambda_0} x(\lambda_0) = x(\lambda_0)$ if $\lambda \to \lambda_0$.
Then the solution $x(\lambda) = T_\lambda x(\lambda)$ is continuous at $\lambda = \lambda_0$.

Proof.
We prove even more:

$$\| x(\lambda) - x(\lambda_0) \| \le \frac{1}{1-q} \| T_\lambda x(\lambda_0) - T_{\lambda_0} x(\lambda_0) \| .$$

Let us consider the equation

$$T_\lambda x = x.$$

The solution x_λ of this equation can be found as a limit of the sequence $\{x_n^\lambda\}$, where $x_{n+1}^\lambda = T_\lambda x_n^\lambda$ with the starting element $x_0^\lambda = x(\lambda_0)$. From the contraction principle we have

$$\| x(\lambda) - x_0^\lambda \| \le \frac{1}{1-q} \| x_0^\lambda - T_\lambda x_0^\lambda \| = \frac{1}{1-q} \| x(\lambda_0) - T_\lambda x(\lambda_0) \| .$$

□

Lemma 2.1. *Let M be a ball in a normed space U:*

$$M = \{u \in U | \quad \| u - u^* \| \le r\},$$

and the operator $T : M \to U$ is a contraction with a constant $0 < q < 1$. If $\| Tu^* - u^* \| \le (1-q)r$, then $T : M \to M$.

Proof.
$\forall u \in M \Rightarrow$ there are the sequence of the inequalities

$$\| Tu - u^* \| \le \| Tu - Tu^* \| + \| Tu^* - u^* \| \le q \| u - u^* \| + (1-q)r \le qr + (1-q)r = r.$$

Thus $T : M \to M$. □

Corollary 2.1. *Let $u = T_\lambda u$ be an equation with a parameter λ and u^* be a solution of this equation for $\lambda = \lambda^*$. Assume that M is the ball in a Banach space U:*

$$M = \{u \in U | \ \| u - u^* \| \le r\},$$

and $\forall\lambda \in V_1$ *the operator* $T_\lambda : M \to U$ *is a contraction. If*

$$T_\lambda u^* \to T_{\lambda^*} u^* = u^* \ \textit{if} \ \ \lambda \to \lambda^*,$$

then there is a neighborhood $V \subset V_1$ *of the point* λ^* *that* $\forall\lambda \in V$, $T_\lambda : M \to M$ *and*

$$u(\lambda) = T_\lambda u(\lambda) \to u(\lambda^*) = u^* \quad \textit{if} \ \ \lambda \to \lambda^*.$$

Remark 2.2. *Note that* $\forall\lambda \in V$ *there exists only solution* $u(\lambda) = T_\lambda u(\lambda)$.

Proof.

Because $T_\lambda u^* \to u^*$ if $\lambda \to \lambda^*$, then there exists a neighborhood V of the point λ^* such that

$$\| T_\lambda u^* - u^* \| \leq (1-q)r.$$

By virtue of the previous lemma one obtains

$$T_\lambda : M \to M, \quad \forall\lambda \in V.$$

From the theorem of continuity with respect to a parameter λ one has the proof of the lemma.
□

2.2 Existence and uniqueness theorems

1. Existence and uniqueness theorems are proven for normal systems of first order ordinary differential equations.

Definition 2.9. *Any system of ODEs with only first derivatives of the form*

$$\frac{dy}{dx} = F(x,y), \quad y \in R^m \tag{2.1}$$

is called a normal system of first order ODEs.

Definition 2.10. *A normal system of ODEs for the unknown functions* $\xi_1(x), \xi_2(x), \ldots, \xi_{m'}(x)$ *is any system of the form*

$$\frac{d^{n(k)}\xi_k}{dx^{n(k)}} = F_k(\xi_1, \frac{d\xi_1}{dx}, \ldots; \xi_2, \frac{d\xi_2}{dx}, \ldots; \xi_{m'}, \frac{d\xi_{m'}}{dx}, \ldots; x) \tag{2.2}$$

$k = \overline{1,m'}$, *in which for each* k *only derivatives* $\frac{d^p\xi_j}{dx^p}$ *of any* ξ_j *of orders* $p < n(j)$ *occur in the right side.*

In other words, the highest derivatives of each function ξ_j can be found only in the left side.

Theorem 2.3. *Every normal system (2.2) of ODEs is equivalent to a first order normal system (2.1).*

Proof.

By introducing new unknown functions:

$$\begin{aligned} y_1 &= \xi_1, y_2 = \frac{d\xi_1}{dx}, \ldots, y_{n(1)} = \frac{d^{n(1)-1}\xi_1}{dx^{n(1)-1}}, \\ y_{n(1)+1} &= \xi_2, y_{n(1)+2} = \frac{d\xi_2}{dx}, \ldots, \\ m &= \sum_{j=1}^{m'} n(j) \end{aligned}$$

we rewrite system (2.2) as

$$\frac{dy_1}{dx} = y_2, \ \frac{dy_2}{dx} = y_3, \ \dots, \ \frac{dy_{n(1)}}{dx} = F_1(y_1, y_2, \dots, y_m; x).$$

The initial value problem for normal system (2.2) is the problem of finding a solution for which the variables

$$\xi_1, \frac{d\xi_1}{dx}, \ \dots, \ \frac{d^{n_1-1}\xi_1}{dx^{n_1-1}}; \ \xi_2, \ \frac{d\xi_2}{dx}, \ \dots; \ \frac{d^{n_2-1}\xi_2}{dx^{n_2-1}}; \ \dots$$

are given at the point $x = x_0$.

Thus, a study of a normal system of first order DEs provides properties for normal systems.

If $m' = 1$ in (2.2), then normal system (2.2) is called a normal equation or an ordinary DE solved with respect to the highest derivative:

$$\frac{d^n y}{dx^n} = F(x, y, y', \dots, y^{(n-1)}).$$

We start studying normal systems from the theorem of existence and uniqueness for a normal first order equations with $m = 1$. For the sake of simplicity, we consider the case with one unknown function, in order to present the main ideas of proofs. All theorems are consequences of the contraction principle. In order to use it one needs to rewrite a normal system of ODEs as an operator equation and to prove that this operator is a contraction.

Let us consider a first order ODE

$$y' = F(x, y), \quad (x, y) \in D \subset R^2,$$

where D is an open set in R^2 and $F \in C(D)$. A point $(x_0, y_0) \in D$ is an arbitrary point in D. Because D is open, then there is the closed rectangle V:

$$V = \{(x, y) \in D \mid |x - x_0| \le a, |y - y_0| \le b\}$$

that $V \subset D$. By virtue of continuity of the function F in D, there is a maximum $m = \max\limits_{(x,y)\in V} |F(x, y)|$. Denote $h = \min(a, \frac{b}{m})$ and $J = [x_0 - h, x_0 + h]$. It is assumed that $m > 0$, because in the case $m = 0$ the existence and uniqueness are simply solved.

Theorem 2.4. *(Picard). Let $F(x, y) \in C(D)$ and satisfy a Lipschitz condition with a constant L in $V \subset D$. Then on the interval J there is only one solution of the Cauchy problem*

$$y' = F(x, y), \quad y(x_0) = y_0, \quad (x_0, y_0) \in D.$$

This solution can be obtained by the iterative method.

Proof.

The Cauchy problem

$$\begin{cases} y' = F(x, y) \\ y(x_0) = y_0 \end{cases}$$

is equivalent to the problem of finding a solution of the operator equation

$$y = Ty,$$

where the operator $T : C(J) \to C(J)$ is defined by the formula

$$Ty = y_0 + \int_{x_0}^{x} F(t, y(t))dt.$$

Let us prove that T is a contraction. For this purpose one has to construct a Banach space U, a closed set $M \subset U$, to prove that $T : M \to M$ and T is a contraction.

Let U be a set of functions $\{u(x) \in C(J)\}$ with the norm

$$\| u \| = \max_{x\in J}(e^{-L|x-x_0|}|u(x)|).$$

This norm is equivalent to the uniform norm on the space of continuous functions $C(J)$:

$$\| u \|_1 = \max_{x\in J} |u(x)|.$$

Exercise 2.8. *Prove that*

$$\| u \| \leq \| u \|_1 \leq e^{Lh} \| u \| .$$

(Hint: $|u(x)| \leq e^{L(h-|x-x_0|)}|u(x)|$ *).*

Theorem 2.5. *(real analysis)* $(C(J), \| \cdot \|_1)$ *is a Banach space.*

Exercise 2.9. *Prove that* $(C(J), \| \cdot \|)$ *is a Banach space.*

Consequence of the exercise is that the space $(C(J), \| \cdot \|)$ is a Banach space.

The set

$$M = \{u(x) \in U | \max_{x\in J} |u(x) - y_0| \leq b\}$$

is a closed set in U.

Exercise 2.10. *Prove that* M *is a closed set in* U*.*

To prove the Picard theorem, firstly, we show that $T : M \to M$. This means that one needs to prove that if $u(x) \in M$, then $Tu \in M$.

Because $u(x) \in C(J)$, then $(Tu)(x) \equiv y_0 + \int_{x_0}^{x} F(t, u(t))dt$ is a continuous function (prove it as exercise). By virtue of the inequalities

$$|y_0 - (Tu)(x)| = |\int_{x_0}^{x} F(t, u(t))dt| \leq |\int_{x_0}^{x} |F(t, u(t))|dt| \leq \int_{x_0}^{x} mdt = m|x - x_0| \leq mh \leq b,$$

one obtains that $T : M \to M$.

The next step is to prove that $T : M \to M$ is a contraction. In order to do this one needs to find a constant q $(0 < q < 1)$ such that $\forall u_1(x) \in M, u_2(x) \in M$:

$$\| Tu_1 - Tu_2 \| < q \| u_1 - u_2 \| .$$

This study is separated into two parts: a) $x \geq x_0$ and b) $x < x_0$.

For the first case $(x \geq x_0)$ one gets the sequence of the inequalities:

$$\begin{aligned}
e^{-L(x-x_0)}|Tu_1 - Tu_2| &\leq e^{-L(x-x_0)} \int_{x_0}^{x} |F(t, u_1(t)) - F(t, u_2(t))|dt \leq \\
&\leq Le^{-L(x-x_0)} \int_{x_0}^{x} |u_1(t) - u_2(t)|dt = \\
&= Le^{-L(x-x_0)} \int_{x_0}^{x} e^{L(t-x_0)}e^{-L(t-x_0)}|u_1(t) - u_2(t)|dt \leq \\
&\leq Le^{-L(x-x_0)} \int_{x_0}^{x} e^{L(t-x_0)} \| u_1 - u_2 \| dt = \\
&= e^{-L(x-x_0)} \| u_1 - u_2 \| \int_{x_0}^{x} e^{L(t-x_0)}d(L(t - x_0)) = \\
&= (1 - e^{-L(x-x_0)}) \| u_1 - u_2 \| \leq \\
&\leq (1 - e^{-Lh}) \| u_1 - u_2 \| .
\end{aligned}$$

The case b) ($x < x_0$) is studied by the same way.

Exercise 2.11. *Prove the inequality*

$$e^{-L(x-x_0)}|Tu_1(x) - Tu_2(x)| \leq (1 - e^{-Lh}) \parallel u_1 - u_2 \parallel, \ \forall x \in J = [x_0 - h, x_0 + h].$$

By taking $q = 1 - e^{-Lh}$ one satisfies the conditions $0 < q < 1$ and

$$\parallel Tu_1 - Tu_2 \parallel \leq q \parallel u_1 - u_2 \parallel, \ \forall u_1, u_2 \in M.$$

Thus, we have constructed the contraction operator $T : M \to M$ with the closed set $M \subset U$, where U is a Banach space.

From the contraction principle we can conclude that there exists $y(x) \in M$ that

$$y(x) = y_0 + \int_{x_0}^{x} F(t, y(t))dt.$$

□

Exercise 2.12. *Using the Picard theorem prove that if*
a) $F(x, y) \subset C(R^2)$,
b) for any rectangle in R^2 there exists L (which depends on the rectangle),
c) there is a constant K that

$$\sup_{(x,y)\in R^2} |F(x, y)| < K.$$

Then for any $(x_0, y_0) \in R^2$ there exists one and only one solution $y(x), \ x \in R^1$.
(Hint: By choosing an arbitrary a one can take b such that $\frac{b}{K} > a$.)

Example 2.5. The Cauchy problem

$$\begin{cases} y' = \sin(x + y^2) \\ y(x_0) = y_0 \end{cases}$$

has a solution for all $x \in R^1$ (apply the previous exercise).

2.2.1 Global theorem

Let us consider the Cauchy problem

$$\begin{cases} y' = F(x, y) \\ y(x_0) = y_0 \end{cases} \tag{2.3}$$

with the function $F(x, y)$, which satisfies the properties:

(a) $F(x, y) \in C(D)$, where $D = \{(x, y) \in R^2 | \ x \in J = (a, b)\}$

(b) the function $F(x, y)$ satisfies a Lipschitz condition in D with the Lipschitz constant $L(x)$, which can depend on x: there is a function $L(x) \in C(J)$ that

$$|F(x, y_1) - F(x, y_2)| \leq L(x)|y_1 - y_2|, \ \forall (x, y_1) \in D, \ (x, y_2) \in D.$$

Remark 2.3. *Left end of the interval J can be equal to $-\infty$ ($a=-\infty$), and right end can also be equal to $+\infty$ ($b=+\infty$).*

Remark 2.4. *It is possible that either $L(x)\to+\infty$ as $x\to a$ or $L(x)\to+\infty$ as $x\to b$.*

Theorem 2.6. *(Global) Let $F(x,y)\in C(D)$ satisfy a Lipschitz condition in D*

$$|F(x,y_1)-F(x,y_2)|\le L(x)|y_1-y_2|,$$

where $D=\{(x,y)\in R^{m+1}|\ \ x\in J=(a,b)\}$. Then there is only one solution of the Cauchy problem

$$y'=F(x,y),\ \ y(x_0)=y_0$$

on the interval J, $\forall(x_0,y_0)\in D$. This solution can be obtained by an iterative method.

Proof.

We use a contraction principle to the operator equation

$$y=Ty$$

where

$$Tu(x)=y_0+\int_{x_0}^{x}F(t,u(t))dt.$$

Here a Banach space is $U=\{u(x)\in C(J)\}$ with the norm

$$\| u \|=\sup_{x\in J}(e^{-\lambda(x)}|u(x)|)<+\infty,$$

where

$$\lambda(x)=\frac{1}{q}|\int_{x_0}^{x}L(t)dt|+|\int_{x_0}^{x}|F(t,y_0)dt||.$$

and the constant q is an arbitrary constant $(0<q<1)$. We will prove that $T:U\to U$ and

$$\| Tu_1-Tu_2 \|\le q \| u_1-u_2 \| . \tag{2.4}$$

In order to prove these properties we consider two cases: (a) $x\ge x_0$ and (b) $x<x_0$.

In case (a) $\lambda(x)=\frac{1}{q}\int_{x_0}^{x}L(t)dt+\int_{x_0}^{x}|F(t,y_0)dt|\ge 0$, thus

$$\lambda'(x)=\frac{1}{q}L(x)+|F(x,y_0)|$$

and, therefore

$$\frac{1}{q}L(x)\le\lambda'(x),\quad |F(x,y_0)|\le\lambda'(x).$$

Any function $u(x)\in U$ satisfies the inequality

$$e^{-\lambda(x)}|Tu(x)|\le e^{-\lambda(x)}|y_0|+e^{-\lambda(x)}\int_{x_0}^{x}|F(t,u(t))|dt.$$

Since

$$|F(t,u)|\le|F(t,y_0)|+|F(t,y_0)-F(t,u)|,$$

one obtains

$$e^{-\lambda(x)}|Tu(x)|\le e^{-\lambda(x)}\left[|y_0|+\int_{x_0}^{x}|F(t,y_0)|dt+\int_{x_0}^{x}|F(t,u(t))-F(t,y_0)|dt\right].$$

Estimating the second and the third terms one has

$$e^{-\lambda(x)}\int_{x_0}^{x}|F(t,y_0)|dt \le e^{-\lambda(x)}\int_{x_0}^{x}\lambda'(t)dt = e^{-\lambda(x)}(\lambda(x)-\lambda(x_0)) = \lambda(x)e^{-\lambda(x)}$$

and

$$e^{-\lambda(x)}\int_{x_0}^{x}|F(t,u(t))-F(t,y_0)|dt \le e^{-\lambda(x)}\int_{x_0}^{x}L(t)(|u(t)|+|y_0|)dt \le$$
$$\le e^{-\lambda(x)}\int_{x_0}^{x}L(t)e^{\lambda(t)}e^{-\lambda(t)}|u(t)|dt + qe^{-\lambda(x)}|y_0|\int_{x_0}^{x}\lambda'(t)dt.$$

Note that the function $f(\lambda)=\lambda e^{-\lambda}$ is bounded. Let $\max_{\lambda}(\lambda e^{-\lambda}) \le C_1$. Since

$$\begin{gathered} e^{-\lambda(x)}\int_{x_0}^{x}L(t)e^{\lambda(t)}e^{-\lambda(t)}|u(t)|dt \le e^{-\lambda(x)} \parallel u \parallel \int_{x_0}^{x}L(t)e^{\lambda(t)}dt \le \\ \le qe^{-\lambda(x)} \parallel u \parallel \int_{x_0}^{x}\lambda'(t)e^{\lambda(t)}dt = \\ = qe^{-\lambda(x)} \parallel u \parallel (e^{\lambda(x)}-e^{\lambda(x_0)}) = q \parallel u \parallel (1-e^{-\lambda(x)}) \le q \parallel u \parallel \end{gathered}$$

and

$$qe^{-\lambda(x)}|y_0|\int_{x_0}^{x}\lambda'(t)dt = q|y_0|e^{-\lambda(x)}\lambda(x) \le q|y_0|\max_{\lambda}(\lambda e^{-\lambda}) \le q|y_0|C_1$$

the value $e^{-\lambda(x)}|Tu(x)|$ is bounded for any $x \ge x_0$.

Exercise 2.13. *Prove the property*

$$|Tu_1 - Tu_2| \le \left(\int_{x_0}^{x}L(t)e^{\lambda(t)}dt\right) \parallel u_1-u_2 \parallel \le e^{\lambda(x)}q \parallel u_1-u_2 \parallel, \quad \forall x \ge x_0.$$

Hint: the proof is the same as in the Picard theorem.
In the case (b) ($x < x_0$):

$$\lambda(x) = \frac{1}{q}\int_{x}^{x_0}L(t)dt + \int_{x}^{x_0}|F(t,y_0)dt|, \quad -\lambda'(x) = \frac{1}{q}L(x) + |F(x,y_0)|.$$

For any $u(x) \in U$ one obtains (by the same way as in the previous case):

$$e^{-\lambda(x)}|Tu(x)| \le e^{-\lambda(x)}[\ |y_0| + \int_{x}^{x_0}|F(t,u(t))|dt + \int_{x}^{x_0}|F(t,u(t))-F(t,y_0)|dt],$$

which means that $e^{-\lambda(x)}|Tu(x)|$ is bounded.

Exercise 2.14. *Prove property (2.4).*
Thus we have proven that $T: U \to U$ and

$$\parallel Tu_1 - Tu_2 \parallel \le q \parallel u_1 - u_2 \parallel .$$

Using the contraction principle one obtains the proof of the theorem. □

Example 2.6. For the equation $y' = e^{x^2}\cos y$, $J = (-\infty, +\infty)$ a Lipschitz constant is $L(x) = e^{x^2}$.

Example 2.7. For the equation $y' = a(x)y + b(x)$, $x \in J$ a Lipschitz constant is $L(x) = |a(x)|$.

Remark 2.5. *If a Lipschitz condition is not satisfied on the whole interval J, then it is possible that there is no solution on the whole interval J.*

For example, the function $F(x,y)$ in the problem

$$y' = y^2, \quad y\left(\frac{1}{2}\right) = -2$$

has no constant L such that

$$|F(u_1) - F(u_2)| = |u_1 + u_2|\ |u_1 - u_2| \leq L|u_1 - u_2|, \quad \forall u_1, u_2.$$

And there is no solution on the intervals $J = (-\infty, \infty)$ or even $J = (-1, 1)$.

Exercise 2.15. *Prove that for the equation*

$$y' = \frac{y \sin^2(e^y)}{1 + y^2}$$

the conditions of the global theorem are not satisfied, but there is one and only one solution on the whole interval $J = (-\infty, \infty)$. Explain why?

(Hint: see the exercise after the Picard theorem).

2.2.2 Existence and uniqueness theorems in the case $m > 1$

Here we study the Cauchy problem for a normal system of first order DE's with m differential equations:

$$\begin{cases} y' = F(x, y) \\ y(x_0) = y_0 \end{cases}$$

where $y = (y_1, y_2, \dots, y_m)$ and $F = (F_1, F_2, \dots, F_m)$.

The following properties of vectors and vector–functions are used.

A norm of the vector y:

$$|y| = \sqrt{\sum_i^m y_i^2}.$$

There are inequalities

1) $|(u, v)| \leq |u||v|, \quad \forall u, v \in R^m$
2) $|u + v| \leq |u| + |v|, \quad \forall u, v \in R^m$
3) $|\int_a^b \vec{u}(t)dt| \leq \int_a^b |\vec{u}(t)|dt, \quad u : [a, b] \to R^m$.

A vector–function $F(x, y)$ satisfies a Lipschitz condition in $D \subset R^{m+1}$ if

$$|F(x, y) - F(x, z)| \leq L|y - z|, \quad \forall (x, y), (x, z) \in D.$$

Remark 2.6. *If D is a convex domain and there are inequalities*

$$|\frac{\partial F_i}{\partial y_j}(x, y)| \leq K, \ \forall i, j = \overline{1, m},$$

then

$$|F(x, u) - F(x, v)| \leq K m^{3/2} |u - v|, \quad \forall (x, u), (x, v) \in D.$$

Let us prove it. If $y(s) = u + s(v - u)$, then from the Lagrange formula there exists $s_* \in [0, 1]$ that

$$F_i(x, v) - F_i(x, u) = F_i(x, y(1)) - F_i(x, y(0)) = \frac{dF_i(x, y(s))}{ds}_{|s=s_*}.$$

By virtue of

$$\frac{dF_i(x, y(s))}{ds} = \sum_j \frac{\partial F_i}{\partial y_j}(x, y(s)) \frac{dy_j(s)}{ds} = \sum_j \frac{\partial F_i}{\partial y_j}(x, y(s))(v_j - u_j)$$

one obtains

$$|F_i(x,v) - F_i(x,u)| \leq \sum_j K|v_j - u_j| \leq \sum_j K|v-u| = mK|v-u|.$$

Therefore

$$|F(x,v) - F(x,u)| = \sqrt{\sum_i (F_i(x,v) - F_i(x,u))^2} \leq \sqrt{\sum_i m^2K^2|v-u|^2} = m^{3/2}K|v-u|.$$

For the norm in the space $C(J)$ one can use the uniform norm

$$\| y(x) \|_1 = \max_{x\in J} |y(x)|,$$

where $J = [a,b]$, $y(x)$ is an arbitrary continuous function.

Theorem 2.7. *(local theorem) Let the normal system of ODE's of the Cauchy problem*

$$y' = F(x,y), \quad y(x_0) = y_0$$

satisfy the properties:
(a) $F(x,y) \in C(D)$, where D is an open set in R^{m+1},
(b) for the cylinder $G = \{(x,y) \in D| \; |x - x_0| \leq a, |y - y_0| \leq b\}$, there are the constants $m = \max_{(x,y)\in G} |F(x,y)|$ and $h = \min(a, \frac{b}{m})$,
(c) $F(x,y)$ satisfies a Lipschitz condition in G.
Then there exists only one solution of the Cauchy problem in the interval $J = [x_0 - h, x_0 + h]$.

Exercise 2.16. *Proof the local theorem.*
Hint: Proof is the same as for the Picard theorem with $m = 1$: define a closed set M in a Banach space U, that $T : M \to M$ is a contraction.

Exercise 2.17. *Formulate a global theorem of existence and uniqueness of a solution for a normal system of first order DE's.*
Hint: see the global theorem for one equation.

Exercise 2.18. *Prove the theorem from the exercise above.*

2.3 Existence without a Lipschitz condition

In this section we show that the existence of solutions of the Cauchy problem may be established without the Lipschitz hypothesis on F. In this case there is no conclusion of uniqueness.

Theorem 2.8. *Suppose that F is continuous in an open domain D. Then for any $(x_0, y_0) \in D$ there exists a solution $y : I \to R^n$ of the Cauchy problem*

$$y' = F(x,y), \quad y(x_0) = y_0, \tag{2.5}$$

defined on some open interval I containing x_0.

The proof of the theorem uses one of the basic results of analysis, known as Ascoli's Theorem or the Ascoli-Arzela Theorem, which we now recall.

Definition 2.11. *Suppose that $S \subset R^p$. A sequence $\{f_m\}_{m=1}^{\infty}$ of functions, $f_m : S \to R^q$, is equicontinuous if for any $\varepsilon > 0$ there is a $\delta > 0$ such that, for any m, and $\forall x, y \in S$ such that $|x - y| < \delta$, then $|f_m(x) - f_m(y)| < \varepsilon$.*

In particular, an equicontinuous sequence is uniformly continuous.

Theorem 2.9. *(Ascoli-Arzela). Let $K \subset R^p$ be compact and let $\{f_m\}$ be an equicontinuous sequence of functions, $f_m : K \to R^q$. Suppose additionally that there is a constant M such that $|f_m(x)| \leq M$ for all m and all $x \in K$. Then there exists a subsequence $\{f_{m_k}\}_{k=1}^{\infty}$ which converges uniformly on K to some limit function $f : K \to R^q$.*

Proof. Let x_i, $i \in N$ be a sequence of points that is dense in K. The sequence $f_m(x_1)$ is bounded; hence it has a convergent subsequence. That is, we can choose a subsequence m_{1j} such that $f_{m_{1j}}(x_1)$ converges as $j \to \infty$. Similar, we can choose a subsequence m_{2j} of the sequence m_{1j} such that $f_{m_{2j}}(x_2)$ converges. Since m_{2j} is a subsequence of m_{1j}, $f_{m_{2j}}(x_1)$ converges as well. Next, we choose a subsequence m_{3j} of the sequence m_{2j} such that $f_{m_{3j}}(x_3)$ converges also at x_3. We proceed in this maaner ad in6nitum. Finally, consider the "diagonal" sequence $f_{m_{jj}}(x)$. Except for the first $i-1$ terms, m_{jj} is a subsequence of m_{ij}; hence $f_{m_{jj}}(x_i)$ converges for every $i \in N$. To simplify notation, we shall set $g_j(x) = f_{m_{jj}}(x)$ in the following.

To conclude the proof, we show that the sequence $g_m(x)$ is uniformly Cauchy. Let $\varepsilon > 0$ be given. The $g_m(x)$, being a subsequence of the $f_m(x)$ are uniformly equicontinuous on K; hence there is a $\delta > 0$ such that $|g_m(y) - g_m(x)| < \varepsilon/3$ whenever $|y - x| < \delta$. Since K is compact, there is a $L \in N$ such that for every $x \in K$ there exists $i \in \{1, ..., L\}$ with $|x_i - x| < \delta$. Now choose P large enough so that $|g_m(x_i) - g_k(x_i)| < \varepsilon/3$ for $m, k > P$ and every $i \in \{1, ..., L\}$. For $m, k > P$ and arbitrary $x \in K$, we now have

$$|g_m(x) - g_k(x)| \leq |g_m(x) - g_m(x_i)| + |g_m(x_i) - g_k(x_i)| + |g_k(x_i) - g_k(x)| < \varepsilon,$$

for some $i \in \{1, ..., L\}$. □

We can now state and prove the fundamental existence result.

Theorem 2.10. *(Peano) Let $V_1 \subset R \times R^n$ be the closed rectangle*

$$V_1 = \{(x, y)| \quad |x - x_0| \leq a, \quad |y - y_0| \leq b\},$$

where $a, b > 0$, and suppose that $F : V_1 \to R^n$ is continuous. Let M be the maximum of $|F|$ on V_1 and let $h = \min\{a, \frac{b}{M}\}$. Then there exists a function $y(x)$ defined on the (closed) interval $J = [x_0 - h, x_0 + h]$ and satisfying the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} F(s, y(s)) \, ds, \tag{2.6}$$

for all $x \in J$.

Proof. We construct $y(x)$ for $x \in J_+ \equiv [x_0, x_0 + h]$. The construction for $x < x_0$ is similar. The method is due to Euler and is frequently mentioned in numerical analysis as a simple scheme to construct approximate solutions of an initial value problem.

For each $m \geq 1$ we subdivide J_+ into m subintervals of the form $[x_{k-1}^{(m)}, x_k^{(m)}]$, where $x_k^{(m)} = x_0 + hk/m$ for $k = 1, \ldots, m$, and construct an approximate solution $y^m(x)$ on J_+ which is linear on each subinterval. The construction is by induction on the index k of the subinterval; we first define $y^m(x_0) = y_0$, and then, assuming that we have constructed y^m with $(x, y^m(x)) \in V_1$ on all intervals $[x_{j-1}^{(m)}, x_j^{(m)}]$ for $j \leq k$, we define

$$y^m(x) = y^m(x_k^{(m)}) + (x - x_k^{(m)}) F(x_k^{(m)}, y^m(x_k^{(m)})), \quad x \in [x_k^{(m)}, x_{k+1}^{(m)}].$$

Note that this definition is chosen so that (i) y^m is continuous at $x_k^{(m)}$, and (ii) on the interval $[x_k^{(m)}, x_{k+1}^{(m)}]$, y^m has derivative $F(x_k^{(m)}, y^m(x_k^{(m)}))$, our best guess at the correct derivative $F(x, y(x))$. In particular, it follows from (i) and (ii) that

$$y^m(x) = y_0 + \int_{x_0}^{x} f^{(m)}(s) \, ds \tag{2.7}$$

for $0 \le x \le x^{(m)}_{k+1}$, where

$$f^{(m)}(t) = F(x^{(m)}_j, y^m(x^{(m)}_j)), \quad t \in [x^{(m)}_j, x^{(m)}_{j+1}].$$

Equation (2.7) implies

$$|y^m(x) - y_0| \le M|x - x_0| \le Mh \le b, \tag{2.8}$$

so that $(x, y^m(x)) \in V_1$ for $x_0 \le x \le x^{(m)}_{k+1}$, allowing us to continue the induction. Eventually we construct y^m on all of J_+.

Now from (2.7) it follows that the sequence $\{y^m\}$ is equicontinuous and uniformly bounded on J_+. In fact, for $x, x' \in J_+$,

$$|y^m(x) - y^m(x')| = |\int_x^{x'} f^{(m)}(s)ds| \le M|x - x'|, \tag{2.9}$$

and

$$|y^m(x)| \le |y^m(x_0)| + |y^m(x) - y^m(x_0)| \le |y_0| + Mh. \tag{2.10}$$

By the Ascoli-Arzela theorem there is thus a subsequence $\{y^{m_j}(x)\}$ which converges uniformly on J_+ to some continuous function $y(x)$. We claim that $y(x)$ is the desired solution of (2.6). This will follow immediately from (2.7) if we can show that $f^{m_j}(x)$ converges uniformly on J_+ to $F(x, y(x))$. It is so, because $y(x)$ is a limit of uniformly convergent and continuous functions $y^{m_j}(x)$. Since F is continuous and hence uniformly continuous on the compact set V_1, the uniform covergence of $y^{m_j}(x)$ to $y(x)$ on J_+ implies the uniform convergence of $F(x, y^{m_j}(x))$ to $F(x, y(x))$ on J_+. Let us check it by the (ε, δ)-language: verify the uniform convergence of the sequence $\{f^{m_j}\}$. For notational simplicity we suppose that the sequence $\{y^m(x)\}$ itself converges to $y(x)$. Then given $\varepsilon > 0$, uniform continuity of F on V_1 implies that there exists a $\delta > 0$ such that $|F(x, y) - F(x', y')| < \varepsilon$ whenever $|(x - x', y - y')| < \delta$. Now choose m so large that $h/m < \delta/3$, that $Mh/m < \delta/3$, and, using the uniformity of convergence of $\{y^m(x)\}$, that $|y^m(x) - y(x)| < \delta/3$ whenever $x \in J_+$. If $x \in J_+$, then $x^{(m)}_k \le x \le x^{(m)}_{k+1}$ for some k, so that by (2.9),

$$|(x^{(m)}_k - x, y^m(x^{(m)}_k) - y(x))| \le |x^{(m)}_k - x| + |y^m(x^{(m)}_k) - y(x)| \le$$
$$\le |x^{(m)}_k - x| + |y^m(x^{(m)}_k) - y^m(x)| + |y^m(x) - y(x)| \le$$
$$\le \frac{h}{m} + \frac{Mh}{m} + \frac{\delta}{3} \le \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta,$$

and hence

$$|f^{(m)}(x) - F(x, y(x))| = |F(x^{(m)}_k, y^m(x^{(m)}_k)) - F(x, y(x))| < \varepsilon.$$

Therefore the function $y(x)$ satisfies the equation

$$y(x) = y_0 + \int_{x_0}^{x} F(s, y(s))\, ds.$$

□

It will in fact be convenient to have at our disposal a formally stronger statement of existence, showing that under appropriate restrictions the interval of definition of the solution may be chosen uniformly in the initial condition, and the pair $(x, y(x))$ may be required to remain in a compact subset of D.

Corollary 2.2. *Let $D \subset R \times R^n$ be open and let $f : D \to R^n$ be continuous. Then for any compact subset $K \subset D$ there exist open sets U, V , with compact closures, satisfying*

$$K \subset U \subset \overline{U} \subset V \subset \overline{V} \subset D, \tag{2.11}$$

and an $\varepsilon > 0$, such that for every $(x_0, y_0) \in U$ a solution $y(x)$ of (2.5) is defined on the interval $I = (x_0 - \varepsilon, x_0 + \varepsilon)$ and satisfies $(x, y(x)) \in \overline{V}$ for $x \in I$.

Proof. Let U, V be any open sets with compact closures satisfying (2.11). Let

$$\eta = d(U, V^c) \equiv \inf_{x \in \overline{U}, y \in V^c} |y - y|,$$

where V^c denotes the complement of V and $d(h, B)$ is the distance between two sets. Because $\overline{U}$ is compact and V^c is closed, η is strictly positive. Let $M = \sup_{\overline{V}} |f(x)|$.

Now if $(x_0, y_0) \in U$, then the rectangle

$$\{(x, y) \mid \ |x - x_0| \le \eta/\sqrt{2},\ |y - y_0| \le \eta/\sqrt{2}\}$$

is contained in $\overline{V}$. The stated result, with $\varepsilon = \min\{\eta/\sqrt{2}, \eta/\sqrt{2}M\}$, now follows from Theorem 2.3, the fundamental existence result proved above. □

2.4 Continuity of solution with respect to initial values and parameters

We study the Cauchy problem

$$y' = F(x, y, \mu), \quad y(x_0) = y_0. \tag{2.12}$$

If the function $F(x, y, \mu)$ has "good" properties (for example, it satisfies the conditions of the existence theorems), then there exists only solution for each fixed x_0, y_0 and parameter μ: $y = \phi(x; \lambda), \quad x \in (a, b)$, where $\lambda = (x_0, y_0, \mu)$ is a vector of parameters.

Definition 2.12. *A solution of the Cauchy problem (2.12) is called continuously dependent with respect to parameters (initial values and some parameters) at the point*

$$x_0 = x_0^*, \quad y_0 = y_0^*, \quad \mu = \mu^*$$

if there exists an interval $[\alpha, \beta] \subset (a, b)$ that $\phi(x, \lambda) \to \phi(x, \lambda^*)$ as $\lambda \to \lambda^*$ uniformly on the interval $[\alpha, \beta]$, (that means in the uniform norm $\| \cdot \|_1$)). Here $\lambda^* = (x_0^*, y_0^*, \mu^*)$.

Exercise 2.19. *Formulate a definition of continuity a solution of the Cauchy problem for an n-th order equation solved with respect to high derivatives.*

We consider the functions $F(x, y, \mu)$ where $(x, y) \in D$, $\mu \in Q$, D is an open set in R^{m+1}, Q is a closed bounded set in R^k. Assume that $F(x, y, \mu) \in C(D \times Q)$ and satisfies a Lipschitz condition in $P \times Q$, i.e. there is a number $L = L(P) > 0$ such that

$$\forall (x, y_1) \in P, (x, y_2) \in P,\ \mu \in Q \Rightarrow |F(x, y_2, \mu) - F(x, y_1, \mu)| \le L|y_1 - y_2|,$$

where P is any closed set in D.

Remark 2.7. *A sufficient condition for satisfying a Lipschitz condition property is $F_y(x, y, \mu) \in C(D \times Q)$, with a convex (with respect to y) domain D.*

Theorem 2.11. *(continuous dependence with respect to parameters) Let the function $F(x, y, \mu)$ satisfy a Lipschitz condition in $P \times Q$:*

$$|F(x, y_1, \mu) - F(x, y_2, \mu)| \le L(P)|y_1 - y_2|, \quad \forall (x, y_1), (x, y_2) \in P,\ \forall \mu \in Q$$

for any closed and bounded $P \subset D$. Then a solution of the Cauchy problem

$$y' = F(x, y, \mu), \quad y(x_0) = y_0$$

is continuous with respect to $\lambda = (x_0, y_0, \mu)$ *at any point* $\lambda^* = (x_0^*, y_0^*, \mu^*) \in D \times Q$.

Here Q is an arbitrary closed bounded set in R^k, P is a closed set in D, D is an open set in R^{m+1}.

Remark 2.8. *Proof of the theorem is based on the abstract theorem for operators* $T_\lambda : M \to U$. *In order to use this theorem we need*

1) to construct a closed ball M in a Banach space U,
2) to show that T_λ is a contraction and

$$T_\lambda u(\lambda^*) \to T_{\lambda^*} u(\lambda^*) = u(\lambda^*) \text{ where } \lambda \to \lambda^*.$$

Proof.
We fix the point

$$\lambda^* = (x_0^*, y_0^*, \mu^*), \quad (x_0^*, y_0^*) \in D, \quad \mu^* \in Q.$$

By virtue of the local theorem of existence and uniqueness there exists a solution $y = \phi(x, \lambda^*)$ of the Cauchy problem in the interval (a, b), $x_0^* \in (a, b)$ and $(x, \phi(x, \lambda^*)) \in D$, $\forall x \in (a, b)$.

We choose a closed and bounded interval $[\alpha, \beta] \subset (a, b)$. Let the operator T_λ be defined by

$$(T_\lambda y)(x) = y_0 + \int_{x_0}^{x} F(t, y(t), \mu) dt.$$

Let us consider the function

$$y_1(x) = \phi(x, \lambda^*) + h e^{L(x-\alpha)}.$$

Lemma 2.2. $\forall L > 0$, $\exists h > 0$ *that* $\forall x \in [\alpha, \beta]$, $\forall s \in [0, 1] \Rightarrow ((1-s)\phi(x, \lambda^*) + s y_1(x), x) \in D$.

Proof (of the lemma).
Assume the opposite assertion, i.e.

$$\exists L > 0, \quad \forall h, \ \exists x \in [\alpha, \beta], \ \exists s \in [0, 1] \Rightarrow ((1-s)\phi(x, \lambda^*) + s y_1(x), x) \notin D.$$

Because h is arbitrary, for example, it can be $h_n = \dfrac{1}{n}$, then there are $x_n \in [\alpha, \beta]$ and $s_n \in [0, 1]$ that $((1 - s_n)\phi(x_n, \lambda^*) + s_n y_1(x_n), x_n) \notin D$. By virtue of the boundedness of $\{x_n\}$ there is a subsequence $x_{n_k} \underset{k\to\infty}{\longrightarrow} x_* \in [\alpha, \beta]$. From continuity of the function $\phi(x, \lambda^*)$ we have that

$$(\phi(x_{n_k}, \lambda^*) + s_{n_k} \frac{1}{n_k} e^{L(x_{n_k} - \alpha)}, x_{n_k}) \underset{k\to\infty}{\longrightarrow} (y_*, x_*).$$

where $y_* = \phi(x_*, \lambda^*)$. Hence, the point $(y_*, x_*) \in D$. For the open set D there is $\delta > 0$ that if $\| (y, x) - (y_*, x_*) \|_2 < \delta$, then $(y, x) \in D$. By constructing the sequence $\{x_{n_k}\}$ there is N that $\forall k > N$

$$\| (\phi(x_{n_k}, \lambda^*) + s_{n_k} \frac{1}{n_k} e^{L(x_{n_k} - \alpha)}, x_{n_k}) - (y_*, x_*) \|_2 < \delta.$$

Therefore, $\forall k > N$ the points $(\phi(x_{n_k}, \lambda^*) + s_{n_k} \frac{1}{n_k} e^{L(x_{n_k} - \alpha)}, x_{n_k})$ belong to D. It contradicts to the assumption that $(\phi(x_{n_k}, \lambda^*) + s_{n_k} \frac{1}{n_k} e^{L(x_{n_k} - \alpha)}, x_{n_k}) \notin D$. The lemma is proved.

Exercise 2.20. *Prove that* $\forall L \ \ \exists h > 0$ *that* $\forall x \in [\alpha, \beta]$, $\forall s \in [0, 1] \Rightarrow$

$$((1-s)\phi(x, \lambda^*) + s y_2(x), x) \in D,$$

where $y_2(x) = \phi(x, \lambda^*) - h e^{L(x-\alpha)}$.

Thus, from the lemma and the exercise we get

$$\forall L > 0, \quad \exists h, \quad \forall x \in [\alpha, \beta], \quad \forall s \in [0, 1] \quad \Rightarrow ((1-s) y_1(x) + s y_2(x), x) \in D.$$

For example, for $L = 1$ there exists h_1 that the strip

$$St_{h_1} = \{(x, y)| \;\; x \in [\alpha, \beta], y_2(x) \le y \le y_1(x)\} \subset D.$$

This strip St_{h_1} is a closed set. By virtue of the condition of the theorem $\exists L(St_{h_1})$ (Lipschitz constant). For $L(St_{h_1})$, $\exists h_2$ that the strip $St_{h_2} \subset D$. We will take $h = \min(h_1, h_2)$. Then the strip $St_h \subset D$ and the function $F(x, y, \mu)$ satisfies a Lipschitz condition on it with the constant $L = L(St_{h_1})$.

The set $U = C([\alpha, \beta])$ with the norm $\| y \| = \max\limits_{x\in[\alpha,\beta]} (e^{-L(x-\alpha)}|y(x)|)$ is a Banach space $(U, \| \cdot \|)$. Denote the ball

$$M = \{y(x) \in U| \; (x, y(x)) \in St_h\}$$

is a closed set in U. We study the operator T_λ:

$$T_\lambda y = y_0 + \int_{x_0}^{x} F(t, y(t), \mu)dt, \quad y \in U.$$

Now we check a satisfaction of the conditions of the theorem for an operator equation with a parameter.

First, $\forall\lambda$ the operators $T_\lambda : M \to U$. Check it as exercise.

Second, $\forall\lambda$ the operators T_λ are contractions with the same number $0 < q < 1$:

$$\| T_\lambda y_1 - T_\lambda y_2 \| \le q \| y_1 - y_2 \|, \quad \forall y_1, y_2 \in M$$

Check it as exercise (hint: proof is the same as the Picard theorem).

Third, $T_\lambda y^* \underset{\lambda\to\lambda^*}{\longrightarrow} T_{\lambda^*} y^* = y^*$. Really,

$$\begin{aligned} & T_\lambda y^* - T_{\lambda^*} y^* = y_0 - y_0^* + \textstyle\int_{x_0}^{x} F(t, y^*(t), \mu)dt - \int_{x_0^*}^{x} F(t, y^*(t), \mu^*)dt \\ = & \;(y_0 - y_0^*) + \textstyle\int_{x_0}^{x} F(t, y^*(t), \mu)dt - (\int_{x_0}^{x} F(t, y^*(t), \mu^*)dt + \int_{x_0^*}^{x_0} F(t, y^*(t), \mu^*)dt) \\ = & \;(y_0 - y_0^*) + \textstyle\int_{x_0}^{x} (F(t, y^*(t), \mu) - F(t, y^*(t), \mu^*))dt + \int_{x_0}^{x_0^*} F(t, y^*(t), \mu^*)dt \\ \equiv & \; I_1 + I_2 + I_3 \end{aligned}$$

where

$$I_1 = y_0 - y_0^*; \quad I_2 = \int_{x_0}^{x} (F(t, y^*(t), \mu) - F(t, y^*(t), \mu^*))dt; \quad I_3 = \int_{x_0}^{x_0^*} F(t, y^*(t), \mu^*)dt.$$

It is simple to prove that $I_1 \underset{\lambda\to\lambda^*}{\longrightarrow} 0$ and $I_3 \underset{\lambda\to\lambda^*}{\longrightarrow} 0$. The last follows from the property that the function $F \in C(D \times Q)$ and $(t, y^*(t)) \in St_h$, where St_h is a closed, bounded set in D. Then there is the maximum

$$R = \max_{(t,y)\in St_h} | \, F(t, y, \mu^*)|$$

By using it we have

$$|I_3| = |\int_{x_0}^{x_0^*} F(t, y^*(t), \mu^*)dt| \le R|x_0 - x_0^*| \underset{\lambda\to\lambda^*}{\longrightarrow} 0.$$

Now we prove that $|I_2| \underset{\lambda\to\lambda^*}{\longrightarrow} 0$. In order to do it we show that

$$\forall\varepsilon > 0, \; \exists\delta > 0, \; \forall\mu, \; |\mu - \mu_*| < \delta, \; \forall(t, y) \in St_h \;\; \Rightarrow |F(t, y, \mu) - F(t, y, \mu_*)| < \varepsilon.$$

Assume the opposite assertion

$$\exists\varepsilon_0, \;\; \forall\delta, \;\; \exists\mu, \;\; |\mu - \mu_*| < \delta, \;\; \exists(t, y) \in St_h \;\; \Rightarrow |F(t, y, \mu) - F(t, y, \mu_*)| \ge \varepsilon_0.$$

For example, δ can be taken as $\delta = \frac{1}{n}$, then we have the sequence of points

$$\{(t_n, y_n, \mu_n)\} \in St_h \times Q$$

in the closed and bounded set $St_h \times Q$. There is the point $(t_*, y_*, \mu_*) \in St_h \times Q$ that $(t_{n_k}, y_{n_k}, \mu_{n_k}) \underset{k\to\infty}{\longrightarrow} (t_*, y_*, \mu_*)$.

Let us consider

$$\begin{aligned} &|F(t_{n_k}, y_{n_k}, \mu_{n_k}) - F(t_{n_k}, y_{n_k}, \mu_*)| \le \\ \le\ & |F(t_{n_k}, y_{n_k}, \mu_{n_k}) - F(t_*, y_*, \mu_*)| + |F(t_*, y_*, \mu_*) - F(t_{n_k}, y_{n_k}, \mu_*)|. \end{aligned}$$

Because $F(x, y, \mu) \in C(D \times Q)$ and $(t_{n_k}, y_{n_k}, \mu_{n_k}) \longrightarrow (t_*, y_*, \mu_*)$, then $\forall\varepsilon > 0,\ \ \exists N,\ \forall k > N$

$$|F(t_{n_k}, y_{n_k}, \mu_{n_k}) - F(t_*, y_*, \mu_*)| < \varepsilon, \quad |F(t_*, y_*, \mu_*) - F(t_{n_k}, y_{n_k}, \mu_*)| < \varepsilon.$$

For example, $\varepsilon = \varepsilon_0/2$. It means that we get the contradiction: at the same time

$$|F(t_{n_k}, y_{n_k}, \mu_{n_k}) - F(t_{n_k}, y_{n_k}, \mu_*)| < \varepsilon_0$$

and

$$|F(t_{n_k}, y_{n_k}, \mu_{n_k}) - F(t_{n_k}, y_{n_k}, \mu_*)| \ge \varepsilon_0.$$

Remark. The property $|I_2| \underset{\lambda\to\lambda^*}{\longrightarrow} 0$ can be easier proven noticing that any continuous function on a closed and bounded set is equicontinuous.

Now we show that $I_3 \underset{\mu\to\mu_*}{\longrightarrow} 0$, i.e.

$$\forall\varepsilon > 0,\ \ \exists\delta > 0,\ \ |\mu - \mu_*| < \delta \ \ \Rightarrow$$

$$\Rightarrow \ \ |I_3| = |\int_{x_0}^{x} (F(t, y^*(t), \mu) - F(t, y^*(t), \mu^*))dt| < \varepsilon.$$

Really, $\forall\varepsilon > 0,\ \ \exists\delta > 0,\ \ \forall\mu,\ \ |\mu - \mu_*| < \delta,\ \ \forall(t, y) \in St_h$ that

$$|F(t, y, \mu) - F(t, y, \mu_*)| < \frac{\varepsilon}{\beta - \alpha}.$$

Substitution it into I_3 gives

$$\begin{aligned} |I_3| \ &\le\ |\int_{x_0}^{x} |F(t, y^*(t), \mu) - F(t, y^*(t), \mu_*)|dt| \\ &\le\ \frac{\varepsilon}{\beta - \alpha}|\int_{x_0}^{x} dt| = \frac{\varepsilon}{\beta - \alpha}|x - x_0| \le \varepsilon. \end{aligned}$$

□

Another an important question is about existence of partial derivatives of the solution $y = \phi(x, \lambda)$ with respect to parameters. We will consider it later.

2.5 Behavior of the solution at the ends of maximal interval

Let us consider a system of ODE's

$$y' = F(x, y).$$

Assume that we have two solutions $y = \phi_1(x),\ x \in (a_1, b_1)$ and $y = \phi_2(x), x \in (a_2, b_2)$, defined in an open domain $D \subset R^{m+1}$.

Definition 2.13. *If the interval* $(a_2, b_2) \subset (a_1, b_1)$ *and*

$$\phi_1(x) = \phi_2(x), \ x \in (a_2, b_2),$$

then the solution $y = \phi_1(x)$ *is called an extension of the solution* $y = \phi_2(x)$, *(in particular, if* $(a_2, b_2) = (a_1, b_1)$*).*

Definition 2.14. *A solution* $y = \phi(x), \ x \in (a, b)$ *is called a nonextendable if there is no its extension excepting itself. A nonextendable solution is also called a maximal solution.*

Theorem 2.12. *(existence of the maximal solution). Let* $F(x, y) \in C(D)$ *and satisfy a Lipschitz condition in* D, *then*

(a) there exists a maximal solution of the Cauchy problem

$$y' = F(x, y), \ \ y(x_0) = y_0, \ \ \forall (x_0, y_0) \in D,$$

the maximal solution is unique;

(b) if a maximal solution of the equations $y' = F(x, y)$ coincides with some solution of these equations at one point, then the maximal solution is an extension of this solution;

(c) if any two maximal solutions coincide at one point, then they have the same intervals, where they are defined and they coincide at every point of this interval.

Proof.

Recall that because of the Lipschitz condition one has uniqueness of solution on the interval where it is defined.

First, we construct a nonextendable solution. Let (x_0, y_0) be any point in D. The Cauchy problem

$$\begin{cases} y' = F(x, y) \\ y(x_0) = y_0 \end{cases} \tag{2.13}$$

has a solution $y = \phi(x), \ x \in J = (s_l, s_r)$. Any solution $y = \phi(x)$ is defined on its own interval J. We will denote by Γ_l a set of the left ends of these intervals and Γ_r is a set of the right ends of the intervals. Let

$$m_l = \inf(\Gamma_l), \quad m_r = \sup(\Gamma_r).$$

We construct the function $\tilde{\phi}(x)$ on the interval (m_l, m_r) by the following way. If x_* is an arbitrary point of the interval (m_l, m_r), then there exists a solution of the Cauchy problem $\phi(x), \ x \in J$ with $J \subset (m_l, m_r)$ and $x_* \in J$. We take $\tilde{\phi}(x_*) = \phi(x_*)$. The value of the function $\tilde{\phi}(x_*)$ does not depend on the solution. Really, let $\phi_1(x), \ x \in J_1$ be another solution of the Cauchy problem (2.13), with the interval which contains the point $x_* \in J_1$. By virtue of uniqueness of the solution of the Cauchy problem, we have $\phi(x_*) = \phi_1(x_*)$. Therefore, the function $\tilde{\phi}(x)$ is uniquely defined for any point $x \in (m_l, m_r)$. The function $\tilde{\phi}(x)$ is a solution of the Cauchy problem (2.13), because in the neighborhood of any point $x_* \in (m_l, m_r)$ it coincides with some solution of the Cauchy problem (2.13).

Let us prove that $\tilde{\phi}(x), \ x \in (m_l, m_r)$ is a nonextendable solution of the Cauchy problem (2.13). Assume that $\phi_1(x), \ x \in (s_l, s_r)$ is an extension of the solution $\tilde{\phi}(x), \ x \in (m_l, m_r)$. Since $\phi_1(x), \ x \in (s_l, s_r)$ is a solution of the Cauchy problem (2.13), then $s_l \geq m_l$ and $m_r \geq s_r$. By virtue of the uniqueness theorem $\phi_1(x) = \tilde{\phi}(x), \ x \in (s_l, s_r)$. This means that $\tilde{\phi}(x)$ is an extension of $\phi_1(x)$.

Assume that there are two nonextandable solutions $\phi_1(x), \ x \in (s_l^{(1)}, s_r^{(1)})$ and $\phi_2(x), \ x \in (s_l^{(2)}, s_r^{(2)})$ of the Cauchy problem (2.13). By virtue of uniqueness $\phi_1(x) = \phi_2(x), \ x \in (s_l^{(1)}, s_r^{(1)}) \cap (s_l^{(2)}, s_r^{(2)})$. If $s_l^{(1)} > s_l^{(2)}$, then the function

$$\phi(x) = \begin{cases} \phi_1(x), & x \in (s_l^{(1)}, s_r^{(1)}) \\ \phi_2(x), & x \in (s_l^{(2)}, s_l^{(1)}) \end{cases}$$

is an extension of the function $\phi_1(x),\ x \in (s_l^{(1)}, s_r^{(1)})$. This contradicts to the assumption that $\phi_1(x),\ x \in (s_l^{(1)}, s_r^{(1)})$ is a nonextendable solution of the Cauchy problem (2.13). Thus, there exists only one nonextendable solution of the Cauchy problem (2.13).

Assume that $\phi(x),\ x \in (m_l, m_r)$ is a maximal (nonextendable) solution of the equations $y' = F(x,y)$, which coincides with some solution of these equations $\phi_1(x),\ x \in (s_l, s_r)$ at a point x_0:

$$\phi(x_0) = \phi_1(x_0).$$

The point x_0 can be chosen as the initial point for the Cauchy problem (2.13) with the value $y(x_0) = \phi_0$. Thus, the maximal solution $\phi(x),\ x \in (m_l, m_r)$ and the solution $\phi_1(x),\ x \in (s_l, s_r)$ are solutions of the same Cauchy problem. As it has proven, then $s_l \geq m_l$ and $m_r \geq s_r$, $\phi(x) = \phi_1(x),\ x \in (s_l, s_r)$. Therefore the solution $\phi(x),\ x \in (m_l, m_r)$ is an extension of the solution $\phi_1(x),\ x \in (s_l, s_r)$.

Now we prove that if $\phi_1(x),\ x \in J_1$ is a nonextendable solution of the equation $y' = F(x,y)$ and $\phi_2(x),\ x \in J_2$ is another nonextendable solution, and at some point $x_0 \in J_1 \cap J_2$ they coincide $\phi_1(x_0) = \phi_2(x_0)$, then $J_1 = J_2$ and $\phi_1(x) = \phi_2(x),\ \forall x \in J = J_1 = J_2$. Really, let $x_0 \in J_1 \cap J_2$ be any point where

$$\phi_1(x_0) = \phi_2(x_0).$$

If one takes x_0 as the initial point for the Cauchy problem, then from the previous it follows that $\phi_1(x)$ is an extension of $\phi_2(x)$ and $\phi_2(x)$ is an extension of $\phi_1(x)$. This means that $J_1 = J_2$ and

$$\phi_1(x) = \phi_2(x), \quad \forall x \in J_1 = J_2.$$

□

The statements of the previous theorem are valid if the Lipschitz condition in an open set D is exchanged with the property that for any closed and bounded set $P \subset D$ the function $F(x,y)$ satisfies a Lipschitz condition in P.

Exercise 2.21. *Prove uniqueness of the Cauchy problem in the case where a function $F(x,y)$, which for any closed bounded set $P \subset D$ the function $F(x,y)$ satisfies a Lipschitz condition in P.*

Theorem 2.13. *(behavior of a maximal solution at the ends). Let $F(x,y) \in C(D)$ with an open set D. Assume that for any closed bounded set $P \subset D$ the function $F(x,y)$ satisfies a Lipschitz condition in P with a Lipschitz constant $L(P)$. Then for any closed bounded set $E \subset D$ and any maximal solution $\phi(x),\ x \in (m_l, m_r)$ there exist s_l and s_r such that $s_l > m_l,\ \ m_r > s_r$ and for all $x \in (m_l, s_l)$ and $x \in (s_r, m_r)$ the point $(x, \phi(x)) \notin E$.*

Proof.

The proof is given for existence of s_r.

In the case $m_r = \infty$, the existence of s_r is trivial. Since the set E is bounded there is $x_* = \max_{(x,y)\in E}\tilde{}(x)$. For any $x \geq x_*$ one has $(x, \phi(x)) \notin E$.

Let m_r be finite ($m_r < \infty$). One has to prove that

$$\exists s_r < m_r, \quad \forall x, \quad s_r < x < m_r \quad \Rightarrow \quad (x, \phi(x)) \notin E.$$

Assume the opposite:

$$\forall s_r < m_r, \quad \exists x, \quad s_r < x < m_r \Rightarrow (x, \phi(x)) \in E.$$

Taking $s_r = m_r - \frac{1}{n}$ one can construct the sequence of points $\{x_n\}$ such that $m_r - \frac{1}{n} < x_n < m_r$ and $(x_n, \phi(x_n)) \in E$. The sequence $\{x_n\}$ is convergent $\{x_n\} \underset{n\to\infty}{\longrightarrow} m_r$. Because E is a closed and bounded set there is a subsequence[1] $\{(x_{n_k}, \phi(x_{n_k}))\} \longrightarrow (m_r, \phi_*)) \in E$. Using the Picard theorem one gets that for the Cauchy problem

$$\begin{cases} y' = F(x,y) \\ y(m_r) = \phi_* \end{cases} \tag{2.14}$$

[1]Notice that a maximal solution is defined in open interval (m_l, m_r).

there exists only one solution $y = \phi_*(x)$, which is defined in the interval $[m_r - h, m_r + h]$, where $h > 0$ is some number. Let us consider the solutions of the Cauchy problems

$$\begin{cases} y' = F(x, y) \\ y(x_{n_k}) = \phi(x_{n_k}) \end{cases} \tag{2.15}$$

Because $(x_{n_k}, \phi(x_{n_k})) \to (m_r, \phi_*)$, then by virtue of the continuity theorem there exists N, such that $\forall k > N$ the solutions $y = \phi_k(x)$ of the Cauchy problems (2.15) are defined in the interval $[m_r - h, m_r + h]$. By virtue of uniqueness $\phi_k(x) = \phi(x), \quad x < m_r$. Therefore, $\forall x < m_r$

$$\lim_{k\to\infty} \phi_k(x) \equiv \phi(x) = \phi_*(x).$$

This means that the function

$$y = \begin{cases} \phi(x), \text{ if } \ x \in (m_l, m_r) \\ \phi_*(x), \text{ if } \ x \in [m_r, m_r + h) \end{cases}$$

is a solution and it is an extension of the solution $\phi(x), \ x \in (m_l, m_r)$. This contradicts the property (to be nonextendable) of a maximal solution. □

Corollary 2.3. *(for autonomous systems). Let $F(y) \in C(D_y)$, where D_y is an open set in R^m. Assume that for any closed bounded set $P_y \subset D_y$ the function $F(y)$ satisfies a Lipschitz condition in P_y with a Lipschitz constant $L(P_y)$. Then for any closed bounded set $E_y \subset D_y$ and any maximal solution $\phi(x), x \in (m_l, m_r)$ with $m_l \neq -\infty$ (or $m_r \neq \infty$) there exists $s_l > m_l$ (or $m_r > s_r$) such that for all $x \in (m_l, s_l)$ (or $x \in (s_r, m_r)$) the point $\phi(x) \notin E_y$.*

Proof.

The proof is given for the case $m_l \neq -\infty$.

Let us define

$$D = \{(x, y) | \ x \in R^1, \ y \in D_y\}.$$

If $P \subset D$ is a closed and bounded set in D, then the set $P_y = \{y \mid \ (x, y) \in P\}$ is closed and bounded in D_y. For any $m \in (m_l, m_r)$ the set $E_m = \{(x, y) | \ x \in [m_l, m], \ y \in E_y\} \subset D$. From the previous theorem there exists $s_l(m) > m_l, \ s_l(m) \in (m_l, m_r)$ such that $(x, \phi(x)) \notin E_m, \ \forall x \in (m_l, s_l(m))$. Let $s_l^*(m) = \min(m, s_l(m))$. Because $(m_l, s_l(m)) \subseteq [m_l, m]$, the statement $(x, \phi(x)) \notin E_m$ is only able for $\phi(x) \notin E_y$. Notice that if $m_1 > m_2$, then $s_l^*(m_1) \geq s_l^*(m_2)$.

Let $s_l = \sup_m(s_l^*(m))$. Since the function $s_l^*(m)$ is nondecreasing, one obtains that $m_l < s_l \leq m_r$. For any $x \in (m_l, s_l)$ there exists m such that $x < s_l^*(m)$. With this m it is proven that $\phi(x) \notin E_y$. □

Corollary 2.4. *Let $F(x, y) \in C(R^{m+1})$, and for any closed bounded set $P \subset R^{m+1}$ the function $F(x, y)$ satisfies a Lipschitz condition in P with a Lipschitz constant $L(P)$. If $\phi(x), x \in (m_l, m_r)$ is a maximal solution with $m_l \neq -\infty$ (or $m_r \neq \infty$), then for $x \to m_l + 0$ (or $x \to m_r - 0$)*

$$|\phi(x)| \to \infty.$$

Exercise 2.22. *Prove the corollary.*

Lemma 2.3. *Let $v(x), x \in (a, b)$ with $v(a) = 0$ satisfies the inequality*

$$v'(x) \leq \alpha + \gamma v(x), \ \gamma \neq 0$$

then

$$v'(x) \leq \alpha e^{\gamma(x-a)}.$$

Proof.

Integrating the inequality

$$\left(v(x)e^{-\gamma(x-a)}\right)' = (v'(x) - \gamma v(x))e^{-\gamma(x-a)} \leq \alpha e^{-\gamma(x-a)},$$

one has

$$v(x)e^{-\gamma(x-a)} \leq -\frac{\alpha}{\gamma}(e^{-\gamma(x-a)} - 1)$$

or

$$v(x) \leq -\frac{\alpha}{\gamma}(1 - e^{-\gamma(x-a)}).$$

Thus,

$$v'(x) \leq \alpha + \gamma v(x) \leq \alpha - \alpha(1 - e^{-\gamma(x-a)}) = \alpha e^{-\gamma(x-a)}.$$

□

Corollary 2.5. *(global theorem). Let $F(x,y) \in C(D)$, where D is an open strip*

$$D = \{(x,y) \in R^{m+1} \mid \ x \in (m_l, m_r), \ y \in R^m\}.$$

Assume that for any closed bounded set $P \subset D$ the function $F(x,y)$ satisfies a Lipschitz condition in P with a Lipschitz constant $L(P)$. If $F(x,y)$ satisfies the inequality

$$|F(x,y)| \leq M(x) + N(x)|y|, \quad \forall (x,y) \in D,$$

where $M(x)$ and $N(x)$ are continuous, non negative functions in (m_l, m_r). Then the unique nonextendable solution of the Cauchy problem

$$\begin{cases} y' = F(x,y) \\ y(x_0) = y_0, \quad (x_0, y_0) \in D \end{cases} \tag{2.16}$$

exists on the entire interval (m_l, m_r).

Proof.

Let $y(x)$, $x \in (\alpha, \beta)$ be the maximal solution of the Cauchy problem (2.16). Note that it is unique and nonextendable. Suppose that $\beta < m_r$. Then there are constants M_1 and N_1 such that $M(x) \leq M_1$ and $N(x) \leq N_1$ $\forall x \in [x_0, \beta]$. Therefore,

$$|y(x)| \leq |y_0| + M_1(\beta - x_0) + N_1 \int_{x_0}^{x} |y(s)| \, ds, \ \forall x \in [x_0, \beta).$$

Thus, choosing $v(x) = \int_{x_0}^{x} |y(s)| \, ds$, and using the lemma, one has

$$|y(x)| \leq (|y_0| + M_1(\beta - x_0))e^{N_1(\beta - x_0)}, \ \forall x \in [x_0, \beta).$$

This means that $y(x)$, $\forall x \in [x_0, \beta)$ remains in a closed bounded set, which contradicts to the previous theorem. Hence, $\beta = m_r$. Similarly, one proves that $\alpha = m_l$. □

2.6 Dynamical systems

Here we study systems of the type

$$\dot{x} = F(x). \tag{2.17}$$

In this section it is assumed that $F(x) \in \mathbf{C}^1(\mathbf{R}^m)$.

Definition 2.15. *An autonomous system (2.17) is called a dynamical system.*

Definition 2.16. *A point $x \in \mathbf{R}^m$ where $F(x) = 0$ is called a critical point of a dynamical system.*

Let $x = \Phi(t),\ \ t \in (a, b)$ be a maximal solution of a dynamical system (2.17). The curve in R^m:

$$l = \{x \mid\ \ x = \Phi(t),\ t \in (a, b)\} \tag{2.18}$$

is called a trajectory of system (2.17) in the phase space R^m.

Theorem 2.14. *Let $\rho(x) \in C^1(R^m)$ and $\rho(x) \neq 0$ for all $x \in R^m$. Then the sets of trajectories of system (2.17) and the system*

$$\dot{x} = \rho(x)F(x) \tag{2.19}$$

coincide.

Proof.

For the sake of simplicity we assume that $\rho > 0$. Let the curve l be a trajectory of system (2.17). Then, by virtue of the definition, there is a solution $x = \Phi(t)$ of system (2.17) that the points of the trajectory are presented by (2.18). In order to prove the theorem we need to find a solution $x = \psi(\lambda),\ \lambda \in (a', b')$ of system (2.19) that the sets l and

$$l' = \{x \mid x = \psi(\lambda),\ \lambda \in (a', b')\}$$

coincide. Assuming that $\psi(\lambda) = \Phi(t(\lambda))$ we obtain:

$$\frac{d\psi(\lambda)}{d\lambda} = \rho(\psi(\lambda))F(\psi(\lambda)) = \frac{d\Phi(t(\lambda))}{dt}\,\frac{dt(\lambda)}{d\lambda} = F(\Phi(t(\lambda)))\,\frac{dt(\lambda)}{d\lambda}.$$

Therefore, for the proof it is enough to find a function $t(\lambda)$ such that

$$\frac{dt(\lambda)}{d\lambda} = \rho(\Phi(t(\lambda))).$$

First of all, we find the interval (a', b') on which the function $t(\lambda)$ is defined. Because $\rho > 0$, then the function

$$\lambda(t) = \int_{t_0}^{t} \frac{1}{\rho(\Phi(\tau))}\,d\tau$$

is a monotonously increasing and continuously differentiable function. The inverse function $t = t(\lambda)$ is a continuously differentiable function on the interval (a', b'), where

$$a' = \lim_{t\to a+0} \lambda(t), \qquad b' = \lim_{t\to b-0} \lambda(t).$$

By the construction of the function

$$\psi(\lambda) = \Phi(t(\lambda)), \quad \lambda \in (a', b')$$

is a solution of the equation (2.19). Therefore for any point x of the trajectory l there exists $t \in (a, b)$ that $x = \Phi(t)$ and there is $\lambda \in (a', b')$ that $x = \Phi(t(\lambda)) = \psi(\lambda)$. $\square$

For studying dynamical systems it is convenient to have the interval (a, b) as the infinite interval $(-\infty, \infty)$. If $F(x) \in C(R^m)$, then $\rho(x) = \dfrac{1}{\sqrt{1+F^2(x)}} \in C(R^m)$ and $\rho(x)F(x)$ is bounded:

$$|\rho(x)F(x)| \leq 1.$$

The solution of the Cauchy problem

$$\dot{x} = \rho(x)F(x), \quad x(t_0) = x_0$$

is defined for any $x_0 \in R^m$ on the infinite interval $(-\infty, \infty)$[2]. Therefore, further we assume for trajectories that they are defined on the infinite interval $(-\infty, \infty)$. If it is not so, then we regard the system

$$\dot{x} = \rho(x)F(x)$$

with $\rho(x) = \dfrac{1}{\sqrt{1+F^2(x)}}$.

Theorem 2.15. *Two trajectories l_1 and l_2 of a dynamical system $\dot{x} = F(x)$ are either not crossed or completely coincide.*

Proof.

Assume that two trajectories

$$\begin{aligned} l_1 &= \{x \mid x = \Phi_1(t),\ t \in R\}, \\ l_2 &= \{x \mid x = \Phi_2(t),\ t \in R\} \end{aligned}$$

have the common point: $\Phi_1(t_1) = \Phi_2(t_2)$. Thus two functions

$$\begin{aligned} \phi_1(t) &= \Phi_1(t),\ t \in R, \\ \phi_2(t) &= \Phi_2(t+c),\ t \in R, \end{aligned}$$

satisfy the same Cauchy problem:

$$\dot{x} = F(x), \quad x(t_1) = \phi_1(t_1) = \Phi_1(t_1) = \Phi_2(t_1 + c) = \phi_2(t_1),$$

where $c = t_2 - t_1$. By virtue of uniqueness of a solution of the Cauchy problem

$$\phi_1(t) = \phi_2(t),\ \forall t \in R.$$

This means that the trajectories l_1 and l_2 coincide. □

Define three classes of the trajectories:

(I) trajectories without selfcrossing: $\Phi(t_1) \neq \Phi(t_2),\ \forall t_1 \neq t_2,$

(II) periodic trajectories: $\exists T > 0$ that $\Phi(t+T) = \Phi(t)$ and

$$\Phi(t_1) \neq \Phi(t_2), \quad \forall t_1, t_2,\ 0 < t_1 < t_2 < T,$$

(III) stationary trajectories: $\Phi(t) = c, \quad \forall t \in R.$

[2] It follows from the exercise about global solution.

Theorem 2.16. *All trajectories are seperated in the three classes: each trajectory belongs to either class (I), or class (II) or class (III).*

Proof.

Assume that a trajectory $x = \Phi(t)$ does not belong to class (I). One can show that in this case it belongs to either class (II) or class (III). In fact, because $\Phi(t) \notin$ (I) there exist t_1, t_2 such that $t_1 < t_2$ and

$$\Phi(t_1) = \Phi(t_2).$$

Let $\tau = t_1 - t_2$. Thus, $\Phi(t)$ and $\Phi(t + \tau)$ coincide at the point $t = t_1$. By virtue of of the theorem of uniqueness of a solution of the Cauchy problem one obtains $\Phi(t) = \Phi(t+\tau), \quad \forall t \in R$. Let K be the set of the numbers τ. Apparently, $0 \in K$, $(t_2 - t_1) \in K$, and also the set K has the properties:

(i) if $\tau \in K$, then $(-\tau) \in K$,

(ii) if $\tau_1 \in K$ and $\tau_2 \in K$, then $(\tau_1 + \tau_2) \in K$.

(iii) the set K is closed.

Let us prove, for example, the third property. Assume that the sequence $\{\tau_n\} \in K$ convrges to some τ. Because

$$\Phi(t) = \Phi(t + \tau_n), \quad \forall t \in R$$

and the function $\Phi(t)$ is a continuous function, then

$$\Phi(t) = \Phi(t + \tau), \quad \forall t \in R.$$

It means that $\tau \in K$ or the set K is closed.

The set

$$K_+ = \{\tau \in K \mid \tau > 0\}$$

is not empty, because, for example, $(t_2 - t_1) \in K_+$. There is

$$T = \inf(K_+) \geq 0.$$

If $T > 0$, then $\Phi(t) \in$ (II), and if $T = 0$, then $\Phi(t) \in$ (III).

Really, assume that $T > 0$. Because K is closed, then $T \in K$. It means, that

$$\Phi(t + T) = \Phi(t), \quad \forall t \in R.$$

By virtue of the property of the infimum

$$\Phi(t_1) \neq \Phi(t_2)$$

for any t_1 and t_2 that

$$0 < t_1 < t_2 < T.$$

Therefore, $\Phi(t) \in$ (II).

Assuming that $T = 0$ one can prove that $K = (-\infty, \infty)$. Let $\tau > 0$ be an arbitrary number. Because $T = 0$, then there exists the sequence $\tau_n \in K_+$ such that $\lim_{n\to\infty} \tau_n = 0$. The numbers

$$Z_n = \tau_n \cdot [\frac{\tau}{\tau_n}]$$

belong to K_+. Here we use the notation $[\ k\ ]$ for an integer part of the number k and $\{\ k\ \}$ for a fractional part of the number k. For example,

$$\frac{\tau}{\tau_n} = [\frac{\tau}{\tau_n}] + \{\frac{\tau}{\tau_n}\}.$$

Therefore

$$\tau = \tau_n[\frac{\tau}{\tau_n}] + \tau_n\{\frac{\tau}{\tau_n}\} = Z_n + \tau_n\{\frac{\tau}{\tau_n}\}$$

Note that $\lim_{n\to\infty} (\tau_n\{\frac{\tau}{\tau_n}\}) = 0$. Since K is closed one obtains

$$\tau = \lim_{n\to\infty} Z_n \ \in K.$$

This means that

$$\Phi(t+\tau) = \Phi(t), \quad \forall t, \ \tau \in (-\infty, \infty).$$

It is possible only for the solution

$$\Phi(t) = c, \ \forall t \in R,$$

where c is a constant. □

2.7 The perturbation equation

Lemma 2.4. *(Hadamard). Let* $g(\lambda, z) \in C(D)$ *and* $\frac{\partial g}{\partial z_j}(\lambda, z) \in C(D)$, $(j = 1, 2, ..., q)$, *where* D *is a convex domain with respect to* $z = (z_1, z_2, ..., z_q)$. *Then there are functions* $h_j(\lambda, z^{(1)}, z^{(2)}) \in C(D_1)$ $(j = 1, 2, ..., q)$ *such that* $h_j(\lambda, z, z) = \frac{\partial g}{\partial z_j}(\lambda, z)$ *and*

$$g(\lambda, z^{(2)}) - g(\lambda, z^{(1)}) = \sum_{j=1}^{q} h_j(\lambda, z^{(1)}, z^{(2)})(z_j^{(2)} - z_j^{(1)}),$$

where $D_1 = \{(\lambda, z^{(1)}, z^{(2)}) \in R^{2q+1} | \ (\lambda, z^{(1)}) \in D, \ (\lambda, z^{(2)}) \in D\}$.

Proof.

For $w(s) = z^{(1)} + s(z^{(2)} - z^{(1)}), \ s \in [0, 1]$ one has

$$g(\lambda, z^{(2)}) - g(\lambda, z^{(1)}) = g(\lambda, w(1)) - g(\lambda, w(0)) = \int_0^1 \frac{\partial}{\partial s} g(\lambda, w(s))ds,$$

where

$$\frac{\partial}{\partial s} g(\lambda, w(s)) = \sum_{j=1}^{q} \frac{\partial g}{\partial z_j}(\lambda, w(s))\frac{dw_j}{ds}(s) = \sum_{j=1}^{q} \frac{\partial g}{\partial z_j}(\lambda, w(s))(z_j^{(2)} - z_j^{(1)}).$$

Therefore,

$$h_j(\lambda, z^{(1)}, z^{(2)}) = \int_0^1 \frac{\partial g}{\partial z_j}(\lambda, w(s))ds, \quad h_j(\lambda, z, z) = \int_0^1 \frac{\partial g}{\partial z_j}(\lambda, z)ds = \frac{\partial g}{\partial x_j}(\lambda, z).$$

Because the functions $\frac{\partial g}{\partial z_j}$ are continuous in D, then $h_j(\lambda, z^{(1)}, z^{(2)}) \in C(D_1)$. □

We study ordinary differential equations with continuous functions $F(x, y, \mu) \in C(D \times Q)$, where (x, y) belongs to an open set $D \subset R^{m+1}$, μ belongs to an open set Q in R^k. Assume that for any closed set $P \subset D$ the function $F(x, y, \mu)$ satisfies a Lipschitz condition in the set $P \times Q$, i.e.

$$\begin{aligned} &\exists L = L(P) > 0, \ \forall (x, y_1) \in P, \ (x, y_2) \in P \text{ and } \ \mu \in Q \Rightarrow \\ &|F(x, y_2, \mu) - F(x, y_1, \mu)| \le L|y_1 - y_2|. \end{aligned} \tag{2.20}$$

Theorem 2.17. *(differentiability with respect to parameters). Let a function* $F(x, y, \mu)$ *satisfies a Lipschitz condition (2.20) in* $P \times Q$ *for any closed and bounded set* $P \subset D$ *and has continuous partial* *derivatives* $\dfrac{\partial F}{\partial \mu_j} \in C(D \times Q)$, $\dfrac{\partial F}{\partial y_l} \in C(D \times Q)$, $(j = 1, 2, ..., k;\ l = 1, 2, ..., m)$. *Then* $\forall (x_0, y_0) \in D$ *there exists unique solution* $\phi(x, \mu)$ *of the Cauchy problem*

$$y' = F(x, y, \mu), \quad y(x_0) = y_0. \tag{2.21}$$

This solution has the following properties:

a) $\phi(x, \mu)$ *is defined in some open set* T *of the space of the variables* (x, μ)*;*

b) $\phi(x, \mu)$ *has continuous derivatives with respect to* μ_j*:* $\dfrac{\partial \varphi}{\partial \mu_j}(x, \mu) \in C(T)$, $(j = 1, 2, ..., k)$, *which are differentiable with respect to* x *and*

$$\frac{\partial^2 \phi}{\partial x \partial \mu_j}(x, \mu) = \frac{\partial^2 \phi}{\partial \mu_j \partial x}(x, \mu) \in C(T), \quad (j = 1, 2, ..., k).$$

Proof.

Let $\mu^* \in Q$ be given and $\phi(x, \mu^*)$, $x \in (x_1, x_2)$ be a maximal solution of the Cauchy problem (2.21) with $\mu = \mu^*$. For any closed interval $[r_1, r_2] \subset (x_1, x_2)$ there exists a number a such that

$$\hat{\Delta} = \{(x, y) |\ \ |y - \phi(x, \mu_*)| \le a,\ x \in [r_1, r_2]\} \subset D.$$

Exercise 2.23. *Prove the existence of* a*.*

Because Q is open, then $\exists b > 0$ such that if $|\mu - \mu^*| \le b$, then $\mu \in Q$. From the continuity theorem (with respect to parameters) there is $\delta > 0$ ($2\delta < b$) that for any μ which satisfies the inequality that $|\mu - \mu^*| < 2\delta$ there exists the solution $\phi(x, \mu)$ of the Cauchy problem (2.21) defined in the interval $[r_1, r_2]$ and satisfies the inequality

$$|\phi(x, \mu) - \phi(x, \mu^*)| < a.$$

Let us define the open set

$$\Delta = \{(x, y, \mu) |\ \ x \in (r_1, r_2),\ |y - \phi(x, \mu^*)| < a,\ |\mu - \mu^*| < 2\delta\}.$$

By the construction $\Delta \subset (D \times Q)$ and it is a convex set with respect to (y, μ).

Let $\mu^{(1)} \in Q$ satisfy the inequality $|\mu^{(1)} - \mu^*| < \delta$. If $|\tau| < \delta$ and

$$\mu^{(2)} = (\mu_1, \mu_2, \ldots, \mu_{l-1}, \mu_l + \tau, \mu_{l+1}, \ldots, \mu_k) = \mu^{(1)} + \tau(0, 0, \ldots, 1, 0, \ldots, 0),$$

then $|\mu^{(2)} - \mu^*| < 2\delta$. Therefore $\forall x \in [r_1, r_2]$ and $\forall \mu_1$ such that $|\mu_1 - \mu^*| < \delta$ one has

$$|\phi(x, \mu^{(1)}) - \phi(x, \mu^*)| < a, \quad |\phi(x, \mu^{(2)}) - \phi(x, \mu^*)| < a.$$

This means that $(x, \phi(x, \mu^{(i)}), \mu^{(i)}) \in \Delta$, $\forall x \in [r_1, r_2]$, $(i = 1, 2)$. Applying the Hadamard lemma one obtains

$$F_j(x, \phi(x, \mu^{(2)}), \mu^{(2)}) - F_j(x, \phi(x, \mu^{(1)}), \mu^{(1)}) =$$

$$= \sum_{\alpha=1}^{m} h^j_\alpha(x, \mu^{(1)}, \mu^{(2)})(\phi_\alpha(x, \mu^{(2)}) - \phi_\alpha(x, \mu^{(1)})) + \sum_{\beta=1}^{k} h^j_{m+\beta}(x, \mu^{(1)}, \mu^{(2)})(\mu^{(2)}_\beta - \mu^{(1)}_\beta).$$

Here $h^j_\alpha = h^j_\alpha(x, \mu^{(1)}, \mu^{(2)})$, $(\alpha = \overline{1, m + k})$ are continuous functions. By the constructon $\mu^{(2)}$ these functions are continuous functions of $(x, \mu^{(1)}, \tau)$. Let us consider the differences

$$\psi_j(x, \mu^{(1)}, \tau) = \frac{\phi_j(x, \mu^{(2)}) - \phi_j(x, \mu^{(1)})}{\tau}, \quad \tau \ne 0.$$

Because $y = \phi(x, \mu^{(1)})$ and $y = \phi(x, \mu^{(2)})$ are solutions

$$\begin{aligned} \frac{d\psi_j(x, \mu^{(1)}, \tau)}{dx} &= \frac{1}{\tau}(F_j(x, \phi(x, \mu^{(2)}), \mu^{(2)}) - F_j(x, \phi(x, \mu^{(1)}), \mu^{(1)})) \\ &= \sum_{\alpha=1}^{m} h^j_\alpha(x, \mu^{(1)}, \tau)\psi_\alpha(x, \mu^{(1)}, \tau) + h^j_{m+l}(x, \mu^{(1)}, \tau). \end{aligned}$$

These relations are valid for any $(x, \mu^{(1)}, \tau)$ such that

$$x \in (r_1, r_2), \ |\mu_1 - \mu_*| < \delta, \quad |\tau| < \delta, \quad \tau \neq 0.$$

Thus, the functions $\psi_j(x, \mu^{(1)}, \tau)$, $(\tau \neq 0)$ satisfy the linear system of ordinary differential equations

$$\frac{dy_j}{dx} = \sum_{\alpha=1}^{m} h^j_\alpha(x, \mu^{(1)}, \tau)y_\alpha + h^j_{m+l}(x, \mu^{(1)}, \tau) \tag{2.22}$$

with the initial data

$$y_j(x_0) = \psi_j(x_0, \mu^{(1)}, \tau) = \frac{\phi_j(x_0, \mu^{(2)}) - \phi_j(x_0, \mu^{(1)})}{\tau} = 0. \tag{2.23}$$

The Cauchy problem (2.22), (2.23) has unique solution $y_j(x) = \chi_j(x, \mu^{(1)}, \tau)$ and this solution is continuous in the set:

$$x \in (r_1, r_2), \quad |\mu^{(1)} - \mu^*| < \delta, \quad |\tau| < \delta.$$

By virtue of the uniqueness one obtains

$$\psi_j(x, \mu^{(1)}, \tau) = \chi_j(x, \mu^{(1)}, \tau), \quad \tau \neq 0.$$

Thus, there exists the limit:

$$\lim_{\tau \to 0} \psi_j(x, \mu^{(1)}, \tau) = \lim_{\tau \to 0} \chi_j(x, \mu^{(1)}, \tau) = \chi_j(x, \mu^{(1)}, 0).$$

By the definition of this limit it is a partial derivative with respect to μ_l:

$$\frac{\partial \phi_j}{\partial \mu_l}(x, \mu^{(1)}) \equiv \lim_{\tau \to 0} \psi_j(x, \mu^{(1)}, \tau) = \chi_j(x, \mu^{(1)}, 0).$$

By virtue of the continuity the functions $\chi_j(x, \mu^{(1)}, 0)$ are continuous functions in the set

$$T = \{(x, \mu) \mid x \in (r_1, r_2), \quad |\mu^{(1)} - \mu^*| < \delta\}. \tag{2.24}$$

Therefore the partial derivatives $\frac{\partial \phi_j}{\partial \mu_l}(x, \mu^{(1)})$ are continuous in this set.

Because $\dfrac{\partial \phi_j}{\partial \mu_l}(x, \mu^{(1)}) = \chi_j(x, \mu^{(1)}, 0)$ and $\chi_j(x, \mu^{(1)}, 0)$ satisfy the Cauchy problem (2.22), (2.23), there is the derivative

$$\frac{d}{dx}(\frac{\partial \phi_j}{\partial \mu_l}(x, \mu^{(1)})) = \frac{d}{dx}\chi_j(x, \mu^{(1)}, 0)$$

which is a continuous function in the set (2.24). There is proven that $\frac{d}{dx}\left(\frac{\partial \phi_j}{\partial \mu_l}(x, \mu^{(1)})\right)$ are continuous functions in the simple connected set (2.24).

Now we prove that the functions $\dfrac{\partial}{\partial \mu_l}(\dfrac{d\phi_j}{dx})(x, \mu^{(1)})$ satisfy the same property. In fact,

$$\frac{d\phi_j}{dx}(x, \mu^{(1)}) = F_j(x, \phi(x, \mu^{(1)}), \mu^{(1)}).$$

Because the functions $F_j(x, y, \mu)$ (right side of this equality) is a continuously differentiable function with respect to y and μ_l, then the functions $F_j(x, \phi(x, \mu^{(1)}), \mu^{(1)})$ are continuously differentiable functions with respect to μ_l (chain rule). This means that the functions

$$\left(\frac{\partial}{\partial \mu_l}(\frac{d\phi_j}{dx})\right)(x, \mu^{(1)})$$

are continuous functions in the set (2.24). Because $(x, \mu^{(1)})$ is an arbitrary point of the set (2.24) and

$$\frac{d}{dx}(\frac{\partial \phi_j}{\partial \mu_l}) = \frac{\partial}{\partial \mu_l}(\frac{d\phi_j}{dx}),$$

then the theorem is proven. □

Remark 2.9. *Here we used the following theorem. If a function $f(x, y)$ has continuous partial derivatives $\frac{\partial^{(2)} f}{\partial x \partial y}(x, y)$ and $\frac{\partial^{(2)} f}{\partial y \partial x}(x, y)$ in a simple connected set D, then*

$$\frac{\partial^{(2)} f}{\partial x \partial y}(x, y) = \frac{\partial^{(2)} f}{\partial y \partial x}(x, y), \quad (x, y) \in D.$$

Remark 2.10. *The Cauchy problem*

$$\begin{cases} \dfrac{dy}{dx} = F(x, y, \mu) \\ y(x_0) = y_0 \end{cases}$$

can be reduced to the equivalent Cauchy problem

$$\begin{cases} \dfrac{dz}{dt} = \hat{F}(t, z, \hat{\mu}) \\ z(0) = 0 \end{cases},$$

where

$$z = y - y_0, \quad t = x - x_0, \quad \hat{F}(t, z, \hat{\mu}) = F(x_0 + t, y_0 + z, \mu).$$

Thus, the initial value (x_0, y_0) can also be considered as parameters.

Chapter 3

Linear systems

3.1 Systems of linear equations

We consider a normal system of n first-order DE's. In the matrix form it can be written as

$$x'(t) = A(t)x(t) + b(t),$$

where $b(t)$ and $x(t)$ are column vectors of the length n.

Theorem 3.1. *If $A(t) \in C(J)$ and $b(t) \in C(J)$, then there exists only one solution of the Cauchy problem:*

$$\begin{cases} x' = Ax + b, \\ x(t_0) = x_0, \ t_0 \in J. \end{cases} \tag{3.1}$$

defined on the whole interval J.

Proof.

For proving the theorem one needs to check conditions of the global theorem. Here $F(t,x) = A(t)x + b(t)$. Thus, $F(t,x) \in C(D)$, where $D = \{(t,x)| \ \ t \in J\}$.For checking the Lipschitz condition in D one has to study

$$F(t,x_1) - F(t,x_2) = A(t)(x_1 - x_2).$$

Therefore $F(t,x)$ satisfies a Lipschitz condition in D with the Lipschitz constant $L(t) = \| A(t) \|_2$. □

3.1.1 Fundamental system of solutions

First of all we consider homogeneous systems ($b \equiv 0$) and establish simple properties of a system of equations

$$\dot{x} = A(t)x. \tag{3.2}$$

Lemma 3.1. *If $x = \varphi(t), \ t \in J$ is a solution of a linear system of ODE's (3.2), which vanishes at some point $t_0 \in J: \ \varphi(t_0) = 0$, then this solution is equal to zero identically in the interval J:*

$$\varphi(t) = 0, \quad \forall t \in J.$$

Proof follows from the theorem of uniqueness of a solution of the Cauchy problem

$$\begin{cases} \dot{x} = A(t)x, \\ x(t_0) = 0. \end{cases}$$

Lemma 3.2. *If $\varphi_1(t), \varphi_2(t), \ldots, \varphi_k(t),\ t \in J$ are solutions of a linear system of ODE's (3.2), then the function*

$$\varphi(t) = \sum_{\alpha=1}^{k} c_\alpha \varphi_\alpha(t), \quad t \in J$$

is a solution of this system. Here c_α, $(\alpha = 1, 2, \ldots, k)$, are arbitrary constants.

Exercise 3.1. *Prove the lemma.*

Definition 3.1. *A system of vector-functions $\varphi_1(t), \varphi_2(t), \ldots, \varphi_k(t),\ t \in J$ is called a linearly dependent system of functions if there are constants $c_\alpha,\ \alpha = 1, 2, \ldots, k$ at least one of them is not equal to zero, such that the linear combination*

$$\sum_{\alpha=1}^{k} c_\alpha \varphi_\alpha(t) = 0,\ \forall t \in J.$$

Otherwise this system of functions is called linearly independent.

From the previous lemmas one can conclude that if at least at one point $t = t_0$ the vectors

$$\varphi_1(t_0),\ \varphi_2(t_0),\ \ldots,\ \varphi_k(t_0)$$

are linearly dependent, then the system of solutions

$$\varphi_1(t),\ \varphi_2(t),\ \ldots,\ \varphi_k(t)$$

is linearly dependent.

Exercise 3.2. *Prove the last property.*

Hint: There are constants c_α such that $\sum_\alpha c_\alpha^2 \neq 0$ and $\sum_\alpha c_\alpha \varphi_\alpha(t_0) = 0$. Study the function $\varphi(t) = \sum c_\alpha \varphi_\alpha(t)$.

Definition 3.2. *A linearly independent system of solutions $\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t),\ t \in J$ is called a fundamental system of solutions (here m is an order of the system).*

Theorem 3.2. *For any linear system of ODE's (3.2), with $A(t) \in C(J)$ there exists a fundamental system of solutions*

$$\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t),\ t \in J.$$

Any solution $\varphi(t),\ t \in J$ of this system can be presented as the sum:

$$\varphi(t) = \sum_{\alpha=1}^{m} c_\alpha \varphi_\alpha(t), \quad \forall t \in J,$$

with some constants $c_\alpha,\ \alpha = 1, 2, \ldots, m$.

Proof.

First of all it is being proven that there exists a fundamental system of solutions of (3.2). Assume that

$$\mathbf{a}_1,\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_m \tag{3.3}$$

is an arbitrary system of linearly independent constant vectors. One constructs a system of solutions $\varphi_j(t)$ by solving the Cauchy problems

$$\begin{cases} \dot{x} = A(t)x, \\ x(t_0) = \mathbf{a}_j. \end{cases}$$

Because a system of vectors (3.3) is linearly independent, then the solutions φ_j, $(j = 1, 2, \ldots, m)$ are independent.

Exercise 3.3. *Prove this.*

By the definition these solutions compose a fundamental system.

For the second part let us consider any solution $\varphi(t)$ of system (3.2). Because the system of vectors (3.3) is linearly independent, then there are constants c_α, $(\alpha = 1, \dots, m)$ such that

$$\varphi(t_0) = \sum_{\alpha=1}^{m} c_\alpha \mathbf{a}_\alpha.$$

The solutions $\varphi(t)$ and $\sum_{\alpha=1}^{m} c_\alpha \varphi_\alpha(t)$ satisfy the same initial value problem and therefore they coincide:

$$\varphi(t) = \sum_{\alpha=1}^{m} c_\alpha \varphi_\alpha(t).$$

□

3.1.2 The Wronskian

Definition 3.3. *Let $\varphi_1(t),\ \varphi_2(t),\ \dots,\ \varphi_m(t),\ t \in J$ be a system of vector-functions. We compose $m \times m$ matrix*

$$(\varphi_1(t),\ \varphi_2(t),\ \dots,\ \varphi_m(t))\,. \tag{3.4}$$

The determinant of this matrix is called the Wronskian:

$$W(t) = \det(\varphi_1(t),\ \varphi_2(t),\ \dots,\ \varphi_m(t)), \quad t \in J.$$

Here $\varphi_j(t)$ is a vector-column

$$\varphi_j(t) = (\varphi_{1j}(t), \varphi_{2j}(t), \dots, \varphi_{mj}(t))^t.$$

If the Wronskian is composed by linearly independent solutions of (3.2), then $W(t) \neq 0$ for any $t \in J$. If the Wronskian $W(t)$ is composed by a linearly dependent system of vectors, then $W(t) = 0,\ \forall t \in J$.

Definition 3.4. *If the Wronskian is composed by a fundamental system of solutions, then the matrix*

$$\Phi(t) = (\varphi_1(t),\ \varphi_2(t),\ \dots,\ \varphi_m(t)), \quad t \in J$$

is called a fundamental matrix of system (3.2).

Theorem 3.3. *Let a matrix*

$$\Phi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_m(t)), \quad t \in J$$

be an arbitrary $m \times m$ matrix with $\varphi_j(t) \in C^1(J),\ (j = 1, 2, \dots, m),\ t \in J$, the determinant of which is not equal to zero for any $t \in J$. Then this matrix is a fundamental matrix only system

$$\dot{x} = A(t)x, \quad A(t) \in C(J).$$

Proof.

Since the matrix $\Phi = (\varphi_{ij}(t))$ composed of vector-columns which are solutions of (3.2), then

$$\dot{\varphi}_{ij}(t) = A_{i\alpha}\varphi_{\alpha_j}(t).$$

Thus,

$$A\Phi = \dot{\Phi}.$$

By virtue of $\det \Phi \neq 0$ there is only one matrix $A = \dot{\Phi}\Phi^{-1}$. Because $\Phi(t) \in C^1(J)$ the matrix $A(t) \in C(J)$. □

3.1.3 The Liouville formula

Theorem 3.4. *(Ostrogradskii-Liouville). Let $W(t)$, $t \in J$ be the Wronskian of a fundamental system of solutions of a linear system of ODE's*

$$\dot{x} = A(t)x, \quad A(t) \in C(J),$$

then

$$W(t) = W(t_0)e^{\int_{t_0}^{t} S(\tau)d\tau}, \ t \in J,$$

where $S(t)$ is a trace of the matrix $A(t)$:

$$S(t) = tr(A(t)) = \sum_{\alpha} A_{\alpha\alpha}(t).$$

Proof.

Let $U = (u_{ij})$ be an $m \times m$ matrix. The determinant $\det U$ of the matrix U can be decomposed

$$\delta_{ij} \det U = u_{i\beta} V_{\beta j},$$

where $V_{\beta j}$ is an algebraic cofactor to the element $u_{j\beta}$. Because the cofactor $V_{\alpha j}$ does not depend on $u_{j\beta}$, then

$$\frac{\partial(\det U)}{\partial u_{j\beta}} = V_{\beta j}.$$

If $u_{\beta j} = \varphi_{\beta j}(t)$, where $\varphi_{\beta j}(t)$ are elements of the matrix $\Phi(t)$, then

$$\frac{d}{dt}(\det \Phi) = \sum_{\alpha,\ j} \frac{\partial(\det \Phi)}{\partial u_{j\alpha}} \dot{\varphi}_{j\alpha} = \sum_{\alpha,\ j} \dot{\varphi}_{j\alpha} V_{\alpha j} = \sum_{j=1}^{m} (\sum_{\alpha=1}^{m} \dot{\varphi}_{j\alpha} V_{\alpha j}) = \sum_{j=1}^{m} \det(\Phi_j).$$

Here the matrices $\Phi_j(t)$ are composed from the matrix $\Phi(t)$, changing j–th row in matrix $\Phi(t)$ by its derivative.

Remark 3.1. *It is obvious that the same formula is valid for the matrixes $\hat{\Phi}_\alpha(t)$, which are obtained from the matrix $\Phi(t)$, changing α–th column by its derivative.*

Assume that $\chi^i(t) = (\varphi_{i1}(t), \varphi_{i2}(t), \ldots, \varphi_{im}(t))$, then

$$\dot{\chi}^i(t) = (\dot{\varphi}_{i1}(t), \dot{\varphi}_{i2}(t), \ldots, \dot{\varphi}_{im}(t)) = (A_{i\alpha}\varphi_{\alpha 1}(t), A_{i\alpha}\varphi_{\alpha 2}(t), \ldots, A_{i\alpha}\varphi_{\alpha m}(t)) =$$

$$= A_{i\alpha}(\varphi_{\alpha 1}(t), \varphi_{\alpha 2}(t), \ldots, \varphi_{\alpha m}(t)) \ = \ A_{i\alpha}\chi^{\alpha}(t).$$

Thus after substituting $\dot{\chi}^i$ in the matrix Φ_i, one has

$$\det \Phi_i = A_{ii} \det(\Phi) = A_{ii} W.$$

Therefore for the Wronskian one obtains the linear equation

$$\dot{W} = \sum_{j=1}^{m} \det \Phi_j = (\sum_{j} A_{jj})W = S(t)W.$$

A solution of this equation is

$$W(t) = W(t_0) \exp(\int_{t_0}^{t} S(\tau)d\tau).$$

□

Remark 3.2. *If $\Phi(t)$ is a fundamental matrix, then the Wronskian is either strictly positive or strictly negative.*

3.1.4 Method of variation of parameters

Any solution of a nonhomogeneous linear system

$$\dot{x}(t) = A(t)x(t) + b(t) \tag{3.5}$$

can be presented as

$$x = x_p + x_h,$$

where x_p is a particular solution of (3.5) and x_h is a solution of a homogeneous linear system of equations.

If a fundamental matrix is known, then a particular solution can be found with the help of quadrature. In fact, let

$$\varphi_1(t), \ldots, \varphi_m(t)$$

be a fundamental system of solutions of a homogeneous system of equations

$$\dot{x} = A(t)x \tag{3.6}$$

We are looking for a solution of (3.5) of the form

$$\varphi(t) = \sum_{\alpha=1}^{m} c_\alpha(t) \cdot \varphi_\alpha(t).$$

Because $\varphi_\alpha(t)$ are solutions of (3.5), then

$$\dot{\varphi} = \sum_{\alpha=1}^{m} c_\alpha \dot{\varphi}_\alpha + \sum_{\alpha=1}^{m} \dot{c}_\alpha \varphi_\alpha = \sum_{\alpha=1}^{m} c_\alpha (A\varphi_\alpha) + \sum_{\alpha=1}^{m} \dot{c}_\alpha \varphi_\alpha.$$

After substituting $\dot{\varphi}$ into (3.5) one gets

$$\dot{\varphi} - (A\varphi + b) = \sum_{\alpha=1}^{m} \dot{c}_\alpha \varphi_\alpha - b = 0.$$

It is a system of ODE's for the functions c_α, which can be rewritten in a matrix form

$$\Phi \dot{c} = b,$$

where Φ is a fundamental matrix, $c(t)$ is the column

$$c(t) = (c_1(t), c_2(t), \ldots, c_m(t))^*.$$

Because $\det(\Phi) \neq 0$, then $\dot{c} = \Phi^{-1} b$ or

$$c(t) = \int \Phi^{-1}(\tau) b(\tau) d\tau = c^0 + \int_{t_0}^{t} \Phi^{-1}(\tau) b(\tau) d\tau$$

Thus a solution $\varphi(t)$ is

$$\varphi(t) = \Phi(t)(c^0 + \int_{t_0}^{t} \Phi^{-1}(\tau) b(\tau) d\tau).$$

In particular, if the matrix $\Phi(t)$ satisfies the condition $\Phi(t_0) = E_m$, where E_m is an identical $m \times m$ matrix (in this case a fundamental matrix Φ is called a matrizant), then a solution of the Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + b(t), \\ x(t_0) = x_0 \end{cases}$$

is

$$x = \Phi(t)(x_0 + \int_{t_0}^{t} \Phi^{-1}(\tau)b(\tau)d\tau).$$

Remark 3.3. *A fundamental matrix satisfies the matrix equation*

$$\dot{\Phi}(t) = A(t)\Phi(t).$$

Therefore, a fundamental matrix is a solution of the equation

$$\dot{X} = A(t)X.$$

Remark 3.4. *For two fundamental matrices $\Phi(t)$ and $\hat{\Phi}(t)$ there is a constant nonsingular matrix P such that*

$$\hat{\Phi}(t) = \Phi(t)P.$$

Exercise 3.4. *Prove the last remark.*

3.2 Periodic linear systems

Definition 3.5. *A linear system of ODE's (3.5) is called a periodic linear system with the period τ if*

$$A(t+\tau) = A(t), \quad b(t+\tau) = b(t), \quad \forall t.$$

Theorem 3.5. *For any fundamental matrix $\Phi(t)$ of a periodic linear system of ODE's with period τ there is a constant nonsingular matrix C such that*

$$\Phi(t+\tau) = \Phi(t)C.$$

Remark 3.5. *The matrix C is called a main matrix.*

Proof.

If a matrix $\Phi(t)$ is a fundamental matrix of a periodic linear system of ODE's, then the matrix $\Phi(t+\tau)$ is a fundamental matrix of the same linear system of ODE's. In fact,

$$\dot{\Phi}(t+\tau) = A(t+\tau)\Phi(t+\tau) = A(t)\Phi(t+\tau).$$

Thus, there is a constant matrix C that

$$\Phi(t+\tau) = \Phi(t)C.$$

□

Theorem 3.6. *If $\Phi(t)$ and $\widehat{\Phi}(t)$ are two fundamental matrices of a periodic linear system of ODE's with a period τ and main matrices C and $\widehat{C}$, then there is a constant matrix P that $\widehat{\Phi}(t) = \Phi(t)P$ and*

$$\widehat{C} = P^{-1}CP.$$

Proof.

Because $\widehat{\Phi}(t)$ and $\Phi(t)$ are two fundamental matrices, then there is a constant matrix P that $\widehat{\Phi}(t) = \Phi(t)P$. It gives

$$\widehat{\Phi}(t+\tau) = \Phi(t+\tau)P = \Phi(t)CP = \widehat{\Phi}(t)P^{-1}CP.$$

□

Definition 3.6. *Two linear periodic systems $\dot{X} = A(t)X$ and $\dot{Y} = \widehat{A}(t)Y$ with the same period τ are called equivalent systems if there is a nonsingular periodic with the period τ matrix $S(t)$ and fundamental matrices $\Phi(t)$ and $\Psi(t)$ (of the systems $\dot{X} = A(t)X$ and $\dot{Y} = \widehat{A}(t)Y$, respectively) such that*

$$\Psi(t) = S(t)\Phi(t).$$

Theorem 3.7. *Two linear periodic systems are equivalent systems if and only if there are fundamental matrices $\Phi(t)$ and $\Psi(t)$ of these systems with the same main matrix C.*

Proof.

Assume that systems

$$\dot{X} = A(t)X \qquad \dot{Y} = \widehat{A}(t)\dot{Y}$$

are equivalent and $\Phi(t)$ is an arbitrary fundamental matrix of the system $\dot{X} = A(t)X$ with the main matrix C. Then $\Psi(t) = S(t)\Phi(t)$ is a fundamental matrix of the system $\dot{Y} = \hat{A}(t)Y$. Therefore,

$$\Psi(t+\tau) = S(t+\tau)\Phi(t+\tau) = S(t)\Phi(t+\tau) = S(t)\Phi(t)C = \Psi(t)C.$$

It proves the first part of the statement of the theorem.

Let two periodic systems with the same period have fundamental matrices with the same main matrix C:

$$\Phi(t+\tau) = \Phi(t)C, \quad \Psi(t+\tau) = \Psi(t)C.$$

Then $C = \Phi^{-1}(t)\Phi(t+\tau)$ and $\Psi(t+\tau) = \Psi(t)\Phi^{-1}(t)\Phi(t+\tau)$ or

$$\Psi(t+\tau)\Phi^{-1}(t+\tau) = \Psi(t)\Phi^{-1}(t).$$

If we denote $S(t) = \Psi(t)\Phi^{-1}(t)$, then $S(t)$ is a periodic matrix and

$$\Psi(t) = S(t)\Phi(t).$$

□

Remark 3.6. *Because linear systems of ODE's are uniquely determined by their fundamental matrices, one can obtain that matrices of equivalent periodic systems are related by*

$$\hat{A}(t) = (\dot{S}(t) + S(t)A(t))S^{-1}(t).$$

3.2.1 Algebraic background

For any nonsingular matrix A there is a matrix B such that

$$AB = BA \text{ and } A = e^B,$$

where
$$e^B = E + B + \frac{1}{2!}B^2 + \ldots + \frac{1}{n!}B^n + \ldots$$

For any real nonsingular matrix A there is a real matrix B_1 that

$$AB_1 = B_1 A \text{ and } A^2 = e^{B_1}.$$

If $AB = BA$, then
$$e^A e^B = e^B e^A = e^{A+B}$$

If
$$Y = e^{tB},$$

then
$$\dot{Y} = BY.$$

In fact, if
$$Y = e^{tB} = E + tB + \frac{1}{2!}t^2B^2 + \ldots + \frac{1}{n!}t^nB^n + \ldots,$$

then

$$\begin{aligned} \dot{Y} &= B + \frac{1}{1!}tB^2 + \frac{1}{2!}t^2B^3 + \ldots + \frac{1}{n!}t^nB^n + \ldots \\ &= B(E + tB + \frac{1}{2!}t^2B^2 + \ldots + \frac{1}{n!}t^nB^n + \ldots) = BY \end{aligned}$$

Exercise 3.5. *Prove that the function $\Psi = e^{tB_1}$ is a fundamental matrix of the system*

$$\dot{Y} = B_1 Y$$

3.2.2 The Liapunov theorem for periodic linear systmes

Theorem 3.8. *(Liapunov). Any periodic linear system of ODE's*

$$\dot{x} = A(t)x, \quad A(t + \tau) = A(t)$$

is equivalent to a linear system
$$\dot{y} = By$$

with the constant matrix B. If a periodic matrix $A(t)$ with period τ is a real matrix then there is a real constant matrix B_1 that these two systems are equivalent systems considering them as periodic systems with the period 2τ.

Proof.

Let a matrix $A(t)$ be a periodic matrix with a period τ, $\Phi(t)$ is a fundamental matrix with the main matrix C:
$$\Phi(t + \tau) = \Phi(t)C.$$

Since C is nonsingular, there exists a matrix B such that $C = e^{\tau B}$. The function $\Psi(t) = e^{tB}$ is a fundamental matrix of the system $\dot{x} = Bx$. Because the periodic linear systems of equations $\dot{x} = A(t)x$ and $\dot{x} = Bx$ have the same main matric C, they are equivalent.

If the matrix $A(t)$ is real, then the fundamental matrix $\Phi(t)$ is a real matrix, this implies that C is also a real matrix and

$$\Phi(t+2\tau) = \Phi(t+\tau)C = \Phi(t)C^2.$$

For the matrix C^2 there is a real matrix $\hat{B}_1$ that

$$C^2 = e^{\hat{B}_1}$$

or if we denote $B_1 = \frac{1}{2\tau}\hat{B}_1$, then $C^2 = e^{2\tau B_1}$ The systems

$$\dot{X} = A(t)X$$

and

$$\dot{Y} = B_1 Y$$

are equivalent systems considering them as systems with the period 2τ. This follows from the property that $\Psi(t) = e^{tB_1}$ is a fundamental matrix of the system $\dot{Y} = B_1 Y$ and the equalities

$$\Psi(t+2\tau) = e^{(t+2\tau)B_1} = e^{tB_1}e^{2\tau B_1} = \Psi(t)C^2, \quad \Phi(t+2\tau) = \Phi(t)C^2.$$

From the previous theorem one has that these two systems are equivalent as periodic systems with the period 2τ. □

3.3 Linear homogeneous systems with constant coefficients

In this subsection we construct a fundamental system of solutions of a linear system of ODE's

$$\dot{x} = Ax, \tag{3.7}$$

where A is a real $m \times m$ matrix with elements $a_{ij} \in R$.

If $x = \phi(t) + i\psi(t)$ is a complex-valued solution of this system, then $\phi(t)$ and $\psi(t)$ are also solutions of (3.7). This follows from the property

$$\dot{\phi} + i\dot{\psi} = A\phi + i(A\psi).$$

A fundamental system of solutions of system (3.7) depends on algebraic structure of the matrix A. Let $D(\lambda) = \det(A - \lambda E)$ be a characteristic polynomial and $\lambda_1, \ldots, \lambda_p$ are eigenvalues of multiplicities $\mu_1, \ldots, \mu_p$. Here $1 \le p \le m$ and $\sum_\alpha \mu_\alpha = m$.

For each eigenvalue λ of multiplicity μ one constructs a chain of adjoint vectors $h^{(1)}, h^{(2)}, \ldots, h^{(\mu)}$:

$$\begin{aligned}
&(A - \lambda E)h^{(1)} = 0 \\
&(A - \lambda E)h^{(2)} = h^{(1)} \\
&\ldots \quad \ldots \\
&(A - \lambda E)h^{(\mu)} = h^{(\mu-1)}.
\end{aligned}$$

Theorem 3.9. *There exists a basis of the vector space $\mathbf{C}^n$, which consists of chains of adjoint vectors. Even more, for real matrices A, if λ is a real number, then a corresponding chain can be chosen real and if λ is a complex number, then for λ and the conjugate eigenvalue $\overline{\lambda}$ corresponding chains can be chosen pairwise conjugate.*

In order to construct a fundamental system of solutions one defines auxiliary functions for each chain.

Let λ be an eigenvalue of multiplicity μ and $h^{(1)}, h^{(2)}, \dots, h^{(\mu)}$ be a chain of adjoint vectors. Functions $\Omega_1(t), \Omega_2(t), \dots, \Omega_\mu(t)$ defined by:

$$\Omega_0(t) = 0, \quad \Omega_k(t) = \sum_{\alpha=1}^{k} \frac{t^{k-\alpha}}{(k-\alpha)!} h^{(\alpha)}.$$

satisfy the properties:

$$\frac{d}{dt}\Omega_k(t) = \Omega_{k-1}(t), \quad (k = 1, 2, \dots, \mu),$$
$$A\Omega_k = \lambda\Omega_k + \Omega_{k-1}, \quad (k = 1, 2, \dots, \mu).$$

Exercise 3.6. *Prove these properties.*

Lemma 3.3. *The vector-function*

$$y^{(k)}(t) = \Omega_k(t)\exp(t\lambda)$$

is a solution of system (3.7).

Exercise 3.7. *Prove this lemma.*

A fundamental system of solutions can be composed by the vector-functions $\Omega_k(t)\exp(\lambda t)$:

$$\Phi(t) = (Re(y^{(1)}),\ Im(y^{(1)}),\ \dots,\ Re(y^{(\mu)}),\ Im(y^{(\mu)}),\ \dots\ ).$$

Exercise 3.8. *Prove that the Wronskian*

$$W(0) = \det(\Phi(0)) \neq 0.$$

3.4 Linear equations of m-th order

The study of the Cauchy problem of any linear equation:

$$y^{(m)} + a_1(t)y^{(m-1)} + \dots + a_m(t)y = b(t), \quad (3.8)$$
$$y(t_0) = y_0,\ y'(t_0) = y_1,\ \dots,\ y^{(m-1)}(t_0) = y_{m-1}, \quad t_0 \in J$$

with coefficients $a_j(t) \in C(J)$ and $b(t) \in C(J)$ can be reduced to the study of the Cauchy problem of the linear system of ODE's

$$x' = A(t)x + \mathbf{b}(t), \quad (3.9)$$
$$x(t_0) = x_0,\ t_0 \in J.$$

Where

$$x_1 = y,\ x_2 = \dot{y},\ \dots,\ x_m = y^{(m-1)}$$

and

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_m(t) & -a_{m-1}(t) & -a_{m-2}(t) & \dots & -a_2(t) & -a_1(t) \end{pmatrix}, \quad \mathbf{b(t)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ b(t) \end{pmatrix}. \quad (3.10)$$

Note that $A(t) \in C(J)$, $\mathbf{b}(t) \in C(J)$.

Theorem 3.10. *The Cauchy problems (3.8) and (3.9) are equivalent.*

Proof.

Any solution $\psi(t)$ of (3.8) corresponds to the solution

$$x(t) = (\psi(t), \psi'(t), \ldots, \psi^{(m-1)}(t))^*$$

of system (3.9), and opposite, any solution of (3.9)

$$x(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t))^*$$

corresponds to the solution

$$y = \varphi_1(t)$$

of (3.8). □

From the equivalence property between (3.8) and (3.9) one concludes that there exists one and only one solution of the Cauchy problem (3.8).

As for a linear system of ODE's we start studying a homogeneous equation

$$y^m + a_1(t)y^{(m-1)} + \ldots + a_m y = 0. \tag{3.11}$$

This equation is equivalent to a homogeneous system

$$\dot{x} = A(t)x$$

with the matrix (3.10).

Definition 3.7. *A system of functions*

$$\psi_1(t), \psi_2(t), \ldots, \psi_k(t), \ t \in J$$

is called a linearly dependent system of functions if there are constants c_α, $(\alpha = 1, 2, \ldots, k)$ at least one of them is not equal to zero, that the linear combination

$$\sum_{\alpha=1}^{k} c_\alpha \psi_\alpha(t) = 0, \ \forall t \in J.$$

Otherwise the system of functions is called linearly independent.

Definition 3.8. *A linearly independent system of solutions*

$$\psi_1(t), \psi_2(t), \ldots, \psi_m(t), \ t \in \ J$$

of a homogeneous equation

$$y^{(m)} + a_1(t)y^{(m-1)} + \ldots + a_m(t)y = 0$$

is called a fundamental system of solutions (here m is an order of the equation).

Theorem 3.11. *A system of solutions $\psi_1(t), \psi_2(t), \ldots, \psi_m(t), \ t \in J$ of equation*

$$y^{(m)} + a_1(t)y^{(m-1)} + \ldots + a_m(t)y = 0$$

is linearly independent (or dependent) if and only if a system of solutions $\vec{\varphi}_1(t), \vec{\varphi}_2(t), \ldots, \vec{\varphi}_m(t), \ t \in J$ of the linear system

$$x' = A(t)x$$

is linearly independent (or linearly dependent).

Remark 3.7. *Here the matrix $A(t)$ is (3.10), the systems of solutions $\psi_1(t), \psi_2(t), \dots, \psi_m(t)$, $t \in J$ and $\vec{\varphi}_1(t), \vec{\varphi}_2(t), \dots, \vec{\varphi}_m(t)$, $t \in J$ are related by the formula*

$$\vec{\varphi}_j(t) = (\psi_j(t), \dot{\psi}_j(t), \dots, \psi_j^{(m-1)}(t))^*$$

Exercise 3.9. *Prove this theorem.*

Definition 3.9. *The determinant of the matrix*

$$W(t) = \begin{vmatrix} \psi_1(t) & \dots & \psi_m(t) \\ \dots & \dots & \dots \\ \psi_1^{(n-1)}(t) & \dots & \psi_m^{(m-1)}(t) \end{vmatrix}, \ t \in J,$$

composed by functions $\psi_1(t), \dots, \psi_m(t)$, $t \in J$ is called the Wronskian.

Theorem 3.12. *(Ostrogradskii-Liouville). Let $W(t)$, $t \in J$ be the Wronskian of a fundamental system of solutions of a linear equation*

$$y^{(m)} + a_1(t)y^{(m-1)} + \dots + a_m(t)y = 0,$$

then

$$W(t) = W(t_0)e^{-\int_{t_0}^t a_1(\tau)d\tau}, \quad t \in J.$$

Exercise 3.10. *Prove the Ostrogradskii-Liouville formula.*

Exercise 3.11. *Derive the method of variation of parameters for equation (3.8).*

3.5 Second order linear equations

The most intensively studied class of ODE's is the class of linear second order equations

$$p_0(x)\frac{d^2u}{dx^2} + p_1(x)\frac{du}{dx} + p_2(x)u = p_3(x). \tag{3.12}$$

The coefficients $p_i(x)$, $(j = 0, 1, 2, 3)$ are assumed continuous functions in the interval J. Dividing (3.12) by leading coefficient $p_0(x)$, one obtains the normal form

$$\frac{d^2u}{dx^2} + p(x)\frac{du}{dx} + q(x)u = r(x). \tag{3.13}$$

By substituting the representation

$$u(x) = v(x)e^{-\frac{1}{2}\int p(x)\ dx}$$

into (3.13) it can be reduced to the equation

$$\frac{d^2v}{dx^2} + \hat{q}(x)v = \hat{r}(x), \tag{3.14}$$

where $\hat{q}(x) = -\dfrac{p'(x)}{2} - \dfrac{p(x)^2}{4} + q(x)$, $\hat{r}(x) = r(x)e^{\frac{1}{2}\int p(x)\ dx}$.

3.5.1 Bases of solutions

Here we consider a second order homogeneous linear equations with constant coefficients

$$y'' + py' + qy = 0.$$

As it was proved, this equation can be transformed to the equation

$$y'' + \hat{q}y = 0.$$

with $\hat{q} = q - p^2/4$. Thus, it is enough to construct a fundamental system of solutions for the equation

$$y'' + qy = 0. \tag{3.15}$$

A fundamental system of solutions $(f(t), g(t))$ of equation (3.15) depends on a value of the constant q:

	$q > 0$	$q = 0$	$q < 0$
$f(t)$	$\sin(t\sqrt{q})$	1	$e^{-t\sqrt{-q}}$
$g(t)$	$\cos(t\sqrt{q})$	t	$e^{t\sqrt{-q}}$

3.5.2 Separation theorems

The following theorem states that all nontrivial solutions of a linear homogeneous second order equation have the same number of zeros.

Theorem 3.13. *(Sturm separation). If $f(t)$ and $g(t)$ are linearly independent solutions of the DE*

$$y'' + p(t)y' + q(t)y = 0,$$

then between two successive zeros of the function $g(t)$ there is zero of the function $f(t)$.

Proof.

Since $f(t)$ and $g(t)$ are linearly independent the Wronskian is either strictly positive or strictly negative. For the sake of simplicity, assume that $W(f,g)(t) > 0$. If $g(t_*)$ vanishes at t_*, then

$$W(f,g)(t_*) = f(t_*)g'(t_*) > 0.$$

Therefore, $f(t_*) \neq 0$ and $g'(t_*) \neq 0$. If t_1 and t_2 are two successive zeros of g: $g(t_i) = 0$, $(i = 1, 2)$, then $g'(t_i) \neq 0$ and $f(t_i) \neq 0$, $(i = 1, 2)$. Moreover, the nonzero numbers $g'(t_i)$, $(i = 1, 2)$ have different signs: $g'(t_1)g'(t_2) < 0$. Hence, $f(t_1)$ and $f(t_2)$ have opposite signs. Because the function $f(t)$ is a continuous function, then there exists a point t_f where $f(t_f) = 0$. □

Theorem 3.14. *(Sturm). Let $f(t)$ and $g(t)$ be nontrivial solutions of the DE's $f'' + p(t)f = 0$, and $g'' + q(t)g = 0$, respectively, where $p(t) \geq q(t)$. Then $f(t)$ vanishes at least once between any two zeros of $g(t)$, unless $p(t) \equiv q(t)$ and $f(t), g(t)$ are linearly dependent.*

Proof.

Let t_1 and t_2 $(t_2 > t_1)$ be two successive zeros of $g(t)$, so that $g(t_i) = 0$, $(i = 1, 2)$ and $g(t) \neq 0$, $\forall t \in (t_1, t_2)$, for example, assume that $g(t) > 0$, $\forall t \in (t_1, t_2)$. Then $g'(t_1) \geq 0$, $g'(t_2) \leq 0$. Suppose that $f(t) \neq 0$, $\forall t \in (t_1, t_2)$, for the sake of simplicity one can account that $f(t) > 0$, $\forall t \in (t_1, t_2)$. This makes

$$W(f,g)(t_1) = f(t_1)g'(t_1) \geq 0, \quad W(f,g)(t_2) = f(t_2)g'(t_2) \leq 0.$$

On the other hand, since $f(t) > 0$, $g(t) > 0$ and $p(t) \geq q(t)$, $\forall t \in (t_1, t_2)$, one has

$$\frac{d}{dt}(W(f,g)(t)) = f(t)g''(t) - f''(t)g(t) = (p(t) - q(t))f(t)g(t) \geq 0, \quad \forall t \in (t_1, t_2).$$

Hence, $W(f,g)(t)$ is nondecreasing. It gives a contradiction unless

$$(p(t)-q(t))f(t)g(t) \equiv W(f,g)(t)) \equiv 0.$$

In this event, $f(t) \equiv kg(t)$ for some constant k. □

Corollary 3.1. *If $q(t) \leq 0$, then there is no nontrivial solution of ODE*

$$u'' + q(t)u = 0 \tag{3.16}$$

that can have more than one zero.

Proof.

We can compare solutions of two equations: (3.16) and $v'' = 0$. If nontrivial solution $g(t)$ of equation (3.16) has two zeros, then the solution $f(t) = 1$ of the equation $v'' = 0$ must have zero between two successive zeros of $g(t)$. This is a contradiction. □

3.5.3 Adjoint operators

Definition 3.10. *Any second–order homogeneous linear ODE*

$$L[u] = p_0(t)u''(t) + p_1(t)u'(t) + p_2(t)u(t) = 0 \tag{3.17}$$

is said to be exact if and only if, for some $A(t), B(t) \in C^1(J)$,

$$p_0u'' + p_1u' + p_2u = \frac{d}{dt}(Au' + Bu)$$

for all functions $u \in C^2(J)$.

An integrating factor for DE (3.17) is a function $v(t)$ such that $vL[u]$ is exact[1].

If an integrating factor v for (3.17) can be found, then

$$v(t)(p_0(t)u'' + p_1(t)u' + p_2(t)u) = \frac{d}{dt}(A(t)u' + B(t)u).$$

Thus, the solutions of the homogeneous DE (3.17) are those of the first–order nonhomogeneous linear DE

$$A(t)u' + B(t)u = C,$$

where C is an arbitrary constant. Also, the solutions of the nonhomogeneous DE $L[u] = r(t)$ are those of the first-order DE

$$A(t)u' + B(t)u = \int v(t)r(t)dt + C$$

These DEs can be solved by a quadrature. Hence, if an integrating factor of (3.17) can be found, one can reduce the equation $L[u] = r(t)$ to a sequence of quadratures.

Evidently, $L[u] = 0$ is exact in (3.17) if and only if $p_0 = A, p_1 = A' + B$ and $p_2 = B'$. Hence (3.17) is exact if and only if

$$p_2 = B' = (p_1 - A')' = p_1' - (p_0')'.$$

This simple calculation proves the following result.

Lemma 3.4. *Differential equation (3.17) is exact if and only if its coefficient functions satisfy*

$$p_0'' - p_1' + p_2 = 0.$$

[1] Here and later, it will be assumed that $p_0 \in C^2(J)$ and $p_1, p_2 \in C^1(J)$.

Corollary 3.2. *A function $v \in C^2(J)$ is an integrating factor for DE (3.17) if and only if it is a solution of the second–order homogeneous linear DE*

$$M[v] = (p_0(t)v(t))'' - (p_1(t)v(t))' + p_2(t)v(t) = 0. \tag{3.18}$$

Definition 3.11. *An operator M in (3.18)*

$$M[v] = p_0 v'' + (2p_0' - p_1)v' + (p_0'' - p_1' + p_2)v = 0$$

is called an adjoint to the linear operator L.

Whenever a nontrivial solution of the adjoint DE (3.18) of a given second–order linear DE (3.17) can be found, every solution of any DE $L[u] = r(t)$ can be obtained by quadratures.

Substituting into (3.18), one finds that an adjoint operatator of the adjoint of a given second–order linear DE is again the original DE. Another consequence of (3.18) is the identity, valid whenever $p_0 \in C^2(J),\ p_i \in C^1(J),\ i = 0, 1,$

$$vL[u] - uM[v] = (vp_0)u'' - u(p_0 v)'' + (vp_1)u' + u(p_1 v)'.$$

Since $wu'' - uw' = (wu' - uw')'$ and $(uw)' = uw' + wu'$, this can be simplified to give the Lagrange identity

$$vL[u] - uM[v] = \frac{d}{dt}(p_0(u'v - uv') - (p_0' - p_1)uv) \tag{3.19}$$

The left side of (3.19) is an exact differential of a homogeneous bilinear expression in $u,\ v$, and their derivatives.

Definition 3.12. *A homogeneous linear differential equation that coincides with its adjoint is called a self-adjoint.*

The condition for (3.17) to be self–adjoint is $2p_0' - p_1 = p_1$, that is $p_0' = p_1$. Since this relation implies $p_0'' = p_1'$, it is also sufficient. Moreover, in this self–adjoint case, the last term in the Lagrange identity (3.19) vanishes. This proves the first statement of the following theorem.

Theorem 3.15. *A second–order linear DE (3.17) is self–adjoint if and only if it has the form*

$$\frac{d}{dt}(p(t)\frac{du}{dt}) + q(t)u = 0.$$

Differential equation (3.17) can be made self–adjoint by multiplying through by

$$h(t) = \frac{1}{p_0(t)} e^{\int (\frac{p_1}{p_0})dt}.$$

Proof.

To prove the second statement, first reduce (3.17) to normal form by dividing through by p_0, and then observe that the DE

$$hu'' + (ph)u' + (qh)u = 0$$

is self–adjoint if and only if $h' = ph$ or $h = e^{\int p dt}$. □

3.5.4 Two–endpoints problem

We have considered only "initial conditions". That is, in considering solutions of second–order DEs such as $y'' = p(t)y' + q(t)y$, we have supposed y and y' both given at the same point t_0. That is natural in many dynamical problems.

In other problems, two–endpoint conditions, at points $t = a$ and $t = b$, are more natural. For instance, the DE $y'' = 0$ characterizes straight lines in the plane, and one may be interested in determining the straight line joining two given points (a, y_1) and (b, y_2). That is, the problem is to find the solution $y = f(t)$ of the DE $y'' = 0$ which satisfies the two–endpoint conditions $f(a) = y_1$ and $f(b) = y_2$.

It is natural to ask: under what circumstances does a second-order DE has an unique solution, assuming given values $f(a) = y_1$ and $f(b) = y_2$ at two given endpoints a and $b > a$? When this is so, the resulting two–endpoint problem is called well-set.

Theorem 3.16. *A two–endpoint problem of a second–order linear homogeneous DE*

$$\begin{aligned} &y'' + p(t)y' + q(t)y = 0, \\ &y(a) = y_1, \quad y(b) = y_2 \end{aligned} \tag{3.20}$$

with continuous coefficients in the interval $J = [a, b]$ has unique solution if and only if the two–endpoint problem

$$\begin{aligned} &y'' + p(t)y' + q(t)y = 0, \\ &y(a) = 0, \quad y(b) = 0 \end{aligned}$$

has an unique solution.

Proof. The general solution of the differential equation $y'' + p(t)y' + q(t)y = 0$ is the function $y = \alpha f(t) + \beta g(t)$, where $f(t)$ and $g(t)$ are two linear independent solutions, and α, β are arbitrary constants. We are looking for the solution that satisfies the conditions

$$\alpha f(a) + \beta g(a) = y_1, \quad \alpha f(b) + \beta g(b) = y_2.$$

Considering this equations as a system of linear algebraic equations with respect to α and β one has that this system has an unique solution α and β if and only if the determinant $f(a)g(b) - g(a)f(b) \neq 0$. This condition coincides with the condition of uniqueness of the solution of the homogeneous linear equations

$$\alpha f(a) + \beta g(a) = 0, \quad \alpha f(b) + \beta g(b) = 0.$$

This proves the theorem. □

When differential equation (3.20) has a nontrivial solution satisfying the homogeneous end–point conditions $y(a) = y(b) = 0$, the point $(b, 0)$ on the x–axis is called a conjugate point of the point $(a, 0)$ for a given homogeneous linear differential equation (3.20). In general, such conjugate points exist for differential equations whose solutions oscillate, but not for those of nonoscillatory type, such as $y'' = q(t)y$ with $q(t) > 0$.

Chapter 4

Stability theory

4.1 Stability definitions

We will consider the normal systems of first–order DE's

$$\dot{x} = F(t, x).$$

with $F(t, x) \in C(D_h)$, where D_h is the cylinder

$$D_h = \{(t, x) \mid \ t > a, |x| < h\}.$$

The constants a and h can accept infinite values: $a = -\infty, \quad h = +\infty$.

If $F(t, x) \in C(D_h)$ and $F_{x_j}(t, x) \in C(D_h)$ $(j = 1, 2, \ldots, m)$, then for any $(t_0, x_0) \in D_h$ there exists only one solution of the Cauchy problem

$$\begin{cases} \dot{x} = F(t, x) \\ x(t_0) = x_0 \end{cases} \tag{4.1}$$

The solution of the Cauchy problem (4.1) will be denoted by $x = \varphi(t, t_0, x_0)$. If it is understandable, then for the solution $x = \varphi(t, t_0, x_0)$ we will write $x = \varphi(t)$. Assuming that $F(t, 0) = 0$ and $x_0 = 0$, we have only solution $x = 0$ of the Cauchy problem (4.1), which is called a trivial solution.

Let us consider some solution $x = \varphi(t)$ of the system $\dot{x} = F(t, x)$ on the interval $(a, +\infty)$. We can construct the equivalent system of equations $\dot{y} = \hat{F}(t, y)$ for which the trivial solution $y = 0$ is equivalent to the solition $x = \varphi(t)$. This can be done by replacing the unknown function $x = y + \varphi(t)$ and setting $\hat{F}(t, y) = F(t, y + \varphi(t)) - F(t, \varphi(t))$. The system $\dot{y} = \hat{F}(t, y)$ is called a reduced system.

Exercise 4.1. *Prove that the systems $\dot{x} = F(t, x)$ and $\dot{y} = \hat{F}(t, y)$ are equivalent.*

Hint: prove that any solution of the system $\dot{x} = F(t, x)$ corresponds to a solution of the system $\dot{y} = \hat{F}(t, y)$ and vice versa.

Definition 4.1. *(Lyapunov stability). The trivial solution of the system $\dot{x} = F(t, x)$ is called a stable solution for $t \to \infty$ if $\forall t_0 \in (a, \infty)$ and $\forall \varepsilon > 0, \ \exists\ \delta(t_0, \varepsilon) > 0$ that any solution $x = \varphi(t, t_0, x_0)$ of the Cauchy problem (4.1) with $|x_0| < \delta$ satisfies the condition $|\varphi(t, t_0, x_0)| < \varepsilon, \quad \forall t \geq t_0$.*

For an arbitrary nontrivial solution $x = \varphi_*(t)$ the following definition of stability can be given.

Definition 4.2. *A solution $x = \varphi_*(t),\ t \in (a, \infty)$ of the system $\dot{x} = F(t, x)$ is called a stable solution if $\forall t_0 \in (a, \infty)$ and $\forall \varepsilon > 0\ \exists\ \delta(t_0, \varepsilon) > 0$ that any solution $x = \varphi(t, t_0, x_0)$ of the Cauchy problem (4.1) with $|x_0 - \varphi_*(t_0)| < \delta$ satisfies the condition $|\varphi(t, t_0, x_0) - \varphi_*(t)| < \varepsilon, \quad \forall t \geq t_0$.*

Solution that is not stable is called unstable.

Thus, the stability of the solution $x = \varphi_*(t_0)$ means that for any point t_0 there is a neighborhood of the point $\varphi_*(t_0)$ such that any solution with initial values from this neighborhood will be close to the solution $x = \varphi_*(t)$, $\forall t > t_0$. And the solution $x = \varphi_*(t)$ is unstable if

$$\exists t_0 \in (a, \infty), \exists \varepsilon > 0, \ \forall \delta > 0 \ \exists x_0, |x_0 - \varphi_*(t_0)| < \delta, \quad \exists t \geq t_0 \ \Rightarrow |\varphi(t, t_0, x_0) - \varphi_*(t)| \geq \varepsilon$$

Definition 4.3. *(uniform stability). A stable solution $x = \varphi_*(t)$ is called uniformly stable if $\delta(t_0, \varepsilon)$ does not depend on t_0:*

$$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0, \ \forall x_0, \ |x_0 - \varphi_*(t_0)| < \delta \Rightarrow |\varphi(t, t_0, x_0) - \varphi_*(t)| < \varepsilon, \quad \forall t > t_0.$$

Exercise 4.2. *Show that the trivial solution $x = 0$ of the equation*

$$\dot{x} + x = 0$$

is stable. Is it a uniformly stable solution?

Definition 4.4. *(asymptotic stability). A stable solution $x = \varphi_*(t)$ is called an asymptoticly stable solution if it possesses the property that $\forall t_0 \in (a, \infty)$ $\exists \triangle(t_0)$ that any solution $x = \varphi(t, t_0, x_0)$ for which $|x_0 - \varphi_*(t_0)| < \triangle(t_0)$ satisfies the equality*

$$\lim_{t \to \infty} |\varphi(t) - \varphi_*(t)| = 0.$$

Note that, in order for a solution to be asymptoticly stable (though not sufficient) condition is to be isolated, i.e. that there is a neighborhood of the solution that does not contain any other asymptoticly stable solutions. This is in contrast to the property of stability, which can apply even to the solutions that are not isolated.

Definition 4.5. *(uniformly asymptotic stability). An asymptoticly stable solution $x = \varphi_*(t)$ is called an uniformly asymptoticly stable solution if there exists $\triangle = \triangle(t_0)$, which does not depend on t_0. The set*

$$\{(t, x) \in R^{m+1} | \ \ t \in (a, \infty), |x - \varphi_*(t)| < \triangle\}$$

is called an attractive domain of the solution $x = \varphi_*(t)$, $t \in (a, \infty)$

Exercise 4.3. *Show that for the equation*

$$\dot{x} = 0$$

any solution is uniformly stable, but not asymptotically stable.

Stability of the solution $x = \varphi_*(t)$ of the system $\dot{x} = F(t, x)$ is equivalent to stability of the trivial solution of the reduced system

$$\dot{y} = \hat{F}(t, y),$$

where $y = x - \varphi_*(t)$ and $\hat{F}(t, y) = F(t, y + \varphi_*(t)) - F(t, \varphi_*(t))$. Later we will consider systems with the trivial solution.

Remark 4.1. *A system*

$$\dot{x} = F(t, x)$$

is periodic with a period τ if $F(t + \tau, x) = F(t, x)$, $\forall t \geq a$, $\forall x \in D$. If the system is autonomous ($F = F(x)$), then it can be thought as a periodic system with an arbitrary period. Hence all the results presented for periodic systems are applied to autonomous systems.

For a periodic system the condition of uniform stability (or uniformly asymptotic stability) is equivalent to stability (asymptotic stability). We leave this remark without proof.

Theorem 4.1. *The trivial solution $x = 0$ of a system $\dot{x} = F(t, x)$ is stable if and only if, $\forall t_0 > a$, $\exists d(t_0)$ and a continuous, strictly increasing function $\phi : R_+ \to R_+$ such that $\phi(0) = 0$ and $\forall x_0$, $|x_0| < d(t_0)$ the solution $x = \varphi(t, t_0, x_0)$ of the Cauchy problem (4.1) satisfies the condition*

$$|\varphi(t, t_0, x_0)| \leq \phi(|x_0|) \ \forall t \geq t_0. \tag{4.2}$$

Proof.
If (4.2) holds, then for any given $\varepsilon > 0$ and any $t_0 \geq a$ by choosing

$$\delta(\varepsilon, t_0) = \begin{cases} d(t_0), & \varepsilon > d(t_0) \\ \phi^{-1}(\varepsilon), & \varepsilon \leq d(t_0) \end{cases}$$

we get that the trivial solution is stable.

To prove the converse, we fix t_0 and construct a function $\phi \in C([0, d(t_0)])$ by the following way.

Because the trivial solution is stable, then for any given $\varepsilon > 0$ there exists $\delta(t_0, \varepsilon)$ such that $\forall x_0$, $|x_0| < \delta(t_0, \varepsilon)$ the solution $x = \varphi(t, t_0, x_0)$ satisfies the inequality $|\varphi(t, t_0, x_0)| < \varepsilon$. The function $\psi_{t_0}(\varepsilon) = \sup(\delta(t_0, \varepsilon))$ is nondecreasing and satisfies the conditions $\psi_{t_0}(0) = 0$ and $\psi_{t_0}(\varepsilon) > 0$, $\forall \varepsilon > 0$. Let $|x_0| < \psi_{t_0}(\varepsilon)$ with $(\varepsilon < h)$, then there is $\delta(t_0, \varepsilon)$, such that $|x_0| < \delta(t_0, \varepsilon) \leq \psi_{t_0}(\varepsilon)$, which implies $|\varphi(t, t_0, x_0)| < \varepsilon$. Since the solution $\varphi(t, t_0, x_0)$ is continuous with respect to x_0, the inequality $|x_0| \leq \psi_{t_0}(\varepsilon)$ implies $|\varphi(t, t_0, x_0)| \leq \varepsilon$.

For any nondecreasing positive function there exists a continuous strictly increasing function $\theta(\varepsilon)$ such that $0 \leq \theta(\varepsilon) \leq \psi_{t_0}(\varepsilon)$, $\forall \varepsilon \geq 0$. Let us define $d(t_0) = \theta(h)$. The function $\theta(\varepsilon)$ has the inverse function $\phi(s) = \theta^{-1}(s)$, which is continuous, strictly increasing and defined in the interval $[0, d(t_0)]$.

Assume that $|x_0| \leq d(t_0)$ and define $\varepsilon = \phi(|x_0|)$. Because $|x_0| = \theta(\varepsilon) \leq \psi_{t_0}(\varepsilon)$, one obtains (4.2).
□

Remark 4.2. *As an example of the function $\theta(\varepsilon)$ one can take the function*

$$\theta(\varepsilon) = \int_0^{\varepsilon} \psi_{t_0}(s) \, ds.$$

Remark 4.3. *Note that the function ϕ depends on the point t_0.*
The same theorem takes place for uniform stability.

Theorem 4.2. *The trivial solution $x = 0$ of a system $\dot{x} = F(t, x)$ is uniformly stable if and only if there exists $d > 0$ and a continuous strictly increasing function $\phi : R_+ \to R_+$ such that $\phi(0) = 0$ and $\forall x_0$, $|x_0| < d$ the solution $x = \varphi(t, t_0, x_0)$ of the Cauchy problem (4.1) satisfies the condition*

$$|\varphi(t, t_0, x_0)| \leq \phi(|x_0|) \ \forall t \geq t_0.$$

Exercise 4.4. *Prove the theorem.*

Definition 4.6. *The function $V(t, x)$ is called a positive definite in the cylinder*

$$D_h = \{(t, x) \mid \ t > a, \ \ |x| \leq h\},$$

if there exists a positive function $W(x) > 0$, $\forall x \neq 0$ such that

$$V(t, x) \geq W(x), \quad \forall (t, x) \in D_h.$$

A function $V(t,x)$ is called a negative definite function in D_h if $-V(t,x)$ is positive definite.

For a given function $V(t,x) \in C^1(D_h)$ and a system $\dot{x} = F(t,x)$ we relate the function

$$\dot{V}(t,x) = \frac{\partial V}{\partial t}(t,x) + \sum_\alpha F_\alpha(t,x)\frac{\partial V}{\partial x_\alpha}(t,x) = \frac{\partial V}{\partial t}(t,x) + \nabla_x V(x)\cdot F(t,x),$$

where $F = (F_1, F_2, \dots, F_m)^*$.

The main property of $\dot{V}(t,x)$ is the following. Let $x = \varphi(t)$ be a solution of the system $\dot{x} = F(t,x)$ and $v(t) = V(t,\varphi(t))$. Then

$$\begin{aligned}\frac{dv(t)}{dt} &= \frac{\partial V}{\partial t}(t,\varphi(t)) + \sum_\alpha \frac{\partial V}{\partial x_\alpha}(t,\varphi(t))\frac{d\varphi_\alpha}{dt}(t) \\ &= \frac{\partial V}{\partial t}(t,\varphi(t)) + \sum_\alpha \frac{\partial V}{\partial x_\alpha}F_\alpha(t,\varphi(t)) \\ &= \dot{V}(t,\varphi(t))\end{aligned}$$

For this reason the function $\dot{V}(t,x)$ is called a derivative of $V(t,x)$ along the trajectories of the system $\dot{x} = F(t,x)$.

Example 4.1. Let $F(t,x) = Ax$ with a constant matrix A and $V(t,x) = x\cdot x = x^2$. For this function

$$\dot{V}(t,x) = 0 + \sum_\alpha \frac{\partial V}{\partial x_\alpha}F_\alpha = 2\sum_\alpha x_\alpha F_\alpha = 2x\cdot F = 2x\cdot Ax.$$

4.2 Stability and instability theorems

4.2.1 The Lyapunov theorems

Theorem 4.3. *(Lyapunov). If there is a positive definite function $V(t,x) \in C^1(D_h)$ with a continuous function $W(x)$ and $V(t,0) = 0$ such that $\dot{V}(t,x) \le 0,\ \forall (t,x) \in D_h$, for a system*

$$\dot{x} = F(t,x),$$

then the trivial solution $x = 0$ of this system is stable.

Remark 4.4. *The function $V(t,x)$ is called a Lyapunov function.*

Proof.

One has to prove that any solution $x = \varphi(t)$ of the system $\dot{x} = F(t,x)$ possesses the property:

$$\forall \varepsilon > 0,\ \ \forall t_0 \in (a,\infty)\ \exists\ \delta = \delta(t_0,\varepsilon) > 0,\ \ \forall |\varphi(t_0)| < \delta \ \Longrightarrow |\varphi(t)| < \varepsilon,\ \ \forall t \in (t_0,\infty).$$

Choose an arbitrary $t_0 \in (a,\infty)$ and $\varepsilon > 0$. Without loss of generality one can suppose that $\varepsilon < h$. Because $W(x)$ is a positive continuous function, then there is the positive number

$$\alpha = \min_{|x|=\varepsilon} W(x) > 0.$$

By virtue of continuity of the function $V(t,x)$ at the point $(t_0,0)$, there exists $\delta > 0$ that if $|x| < \delta$, then $V(t_0,x) < \alpha$. It is also assumed that $\delta < \varepsilon$.

Now we will show that if $x = \varphi(t,t_0,x_0)$ is any solution of the Cauchy problem (4.1) and if $|x_0| < \delta$, then $|\varphi(t,t_0,x_0)| < \varepsilon,\ \ \forall t > t_0$.

Assume that there is t_1 such that $|\varphi(t_1, t_0, x_0)| = \varepsilon$ and t_1 is the nearest point to t_0 with this property. The function $v(t) = V(t, \varphi(t)) \geq W(\varphi(t))$ is not negative and satisfies the inequality

$$v(t_1) - v(t_0) = \int_{t_0}^{t_1} \dot{V}(\tau, \varphi(\tau))d\tau \leq 0.$$

This means that

$$v(t_0) \geq v(t_1).$$

Note that $v(t_1) > 0$, because of $\varphi(t_1) \neq 0$. But from another side, because $|\varphi(t_0)| < \delta$, then

$$v(t_0) = V(t_0, \varphi(t_0)) < \alpha,$$

This gives

$$v(t_1) = V(t_1, \varphi(t_1)) \geq W(\varphi(t_1)) \geq \min_{|x|=\varepsilon} W(x) = \alpha > v(t_0).$$

This contradicts to the assumption. □

Exercise 4.5. *Prove that the trivial solution of the equation*

$$\dot{x} = ax, \quad a \leq 0$$

is stable. Use the Lyapunov function $V(t, x) = x^2$.

Definition 4.7. *A function* $V(t, x)$ *is said to be a decrescent if there exists a constant* $h > 0$ *and a continuous strictly increasing function* $W(s)$, $s \in [0, h]$ *such that*

i) $W(0) = 0$,

ii) $V(t, x) \leq W(|x|), \quad \forall t \in (a, \infty), \quad \forall x \in B_h$, where $B_h = \{x \in R^n | \ \ |x| < h\}$.

Theorem 4.4. *(Lyapunov, asymptotic stability). The equilibrium* $x = 0$ *of the system* $\dot{x} = F(t, x)$ *is asymptotically stable if there exists a decrescent, positive definite function* $V(t, x) \in C^1(D_h)$ *such that*

$$V(t, x) \geq W(x), \forall t \in (a, \infty), \forall x \in B_h,$$

$$V(t, x) \leq W_2(|x|), \quad \forall t \in (a, \infty), \quad \forall x \in B_h.$$

and $\dot{V}(t, x)$ *is a negative definite function:*

$$\dot{V}(t, x) \leq -W_1(x) < 0, \ \forall x \neq 0.$$

Here $W_2(s) \in C([0, h])$ *is a monotonously increasing function,* $W(x) \in C(B_h)$, $W_1(x) \in C(B_h)$, *and*

$$V(t, 0) = 0, \quad W_2(0) = 0.$$

Proof.
Since

$$\dot{V}(t, x) \leq -W_1(x) < 0, \ \forall x \neq 0,$$

by virtue of the first Lyapunov theorem the trivial solution is stable. Hence, taking $\varepsilon = h$, there exists $\delta(t_0, h) > 0$ such that $\forall x_0, \ |x_0|, \delta$ one has $|\varphi(t, t_0, x_0)| < h$. Set $\Delta(t_0) = \delta(t_0, h)$. Let $|x_0| < \Delta(t_0)$ and assume that $x = \varphi(t, t_0, x_0)$ is the solution of the Cauchy problem

$$\begin{cases} \dot{x} = F(t, x), \\ x(t_0) = x_0, \end{cases}$$

where $|x_0| < \Delta(t_0),\ t_0 > a$. First of all we study the function

$$v(t) = V(t, \varphi(t, t_0, x_0)).$$

If $v(t_0) = 0$, then

$$0 = V(t_0, \varphi(t_0)) \geq W(\varphi(t_0)) = W(x_0) \geq 0.$$

Therefore $W(x_0) = 0$, but it can be only if $x_0 = 0$. In this case $\varphi(t, t_0, x_0) = 0,\ \ \forall t \geq t_0$ (by virtue of uniqueness of a solution of the Cauchy problem).

Let $v(t_0) \neq 0$. Because $\varphi(t, t_0, x_0) \in B_h$ and

$$\dot{v}(t) = \dot{V}(t, \varphi(t, t_0, x_0)) \leq -W_1(\varphi(t, t_0, x_0)) < 0,$$

the function $v(t)$ is a monotonously decreasing function. Therefore there is a limit of $v(t)$:

$$\alpha = \lim_{t\to\infty} v(t) \geq 0.$$

We will show that $\alpha = 0$.

Assume that $\alpha > 0$, then the solution $x = \varphi(t, t_0, x_0)$ has the property:

$$\exists\ \beta > 0 \Longrightarrow |\varphi(t)| \geq \beta, \quad \forall t > t_0.$$

In fact, suppose opposite: $\forall\ \beta > 0, \quad \exists\ t > t_0$ that $|\varphi(t)| < \beta$. Choosing $\beta_k = 1/k$, one has the sequence $(t_k,\ \varphi(t_k))$ in which $\varphi(t_k) \to 0$. If $\{t_k\}$ is bounded, then there is a subsequence

$$\{t_{k_n}\} \underset{n\to\infty}{\longrightarrow} \hat{t}_0 < \infty.$$

Because of the continuity of the function $\varphi(t)$

$$\varphi(\hat{t}_0) = \lim_{n\to\infty} \varphi(t_{k_n}) = 0.$$

By virtue of the uniqueness of a solution of the Cauchy problem (4.1) one obtains that $\varphi(t, t_0, x_0) = 0$, $\forall t > t_0$. This gives a contradiction to $v(t_0) \neq 0$. Therefore there exists subsequence $t_{k_n} \to \infty$.

Because the function $V(t, x)$ is decrescent, then there exists a function $W_2(x)$ such that $W_2(0) = 0$ and

$$V(t, x) \leq W_2(|x|), \quad \forall t \in (a, \infty), \quad \forall x \in B_h.$$

Because $\varphi(t, t_0, x_0) \in B_h$ and $\varphi(t_{k_n}, t_0, x_0) \to 0$, then

$$0 \leq v(t_{k_n}) = V(t_{k_n}, \varphi(t_{k_n}, t_0, x_0)) \leq W_2(|\varphi(t_{k_n})|) \to 0.$$

It contradicts to the condition that $\alpha > 0$. Thus,

$$\exists\ \beta > 0 \Longrightarrow |\varphi(t)| \geq \beta, \quad \forall t > t_0.$$

Let us consider the ring

$$G = \{x|\ \ \beta \leq x \leq h\}.$$

Because the function $W_1(x)$ is continuous and positive, then there is a strictly positive constant $\gamma = \min_{x\in G} W_1(x) > 0$. This implies

$$v(t) - v(t_0) \leq \int_{t_0}^{t} W_1(\varphi(s, t_0, x_0))\, ds \leq -\gamma(t - t_0).$$

It contradicts to the condition that $v(t) \geq 0$. Therefore $\alpha = 0$.

By virtue of the inequality

$$v(t) = V(t, \varphi(t, t_0, x_0)) \geq W(\varphi(t, t_0, x_0)) > 0$$

one obtains

$$\lim_{t \to \infty} W(\varphi(t, t_0, x_0)) = 0.$$

It means that

$$\lim_{t \to \infty} \varphi(t, t_0, x_0) = 0.$$

□

Exercise 4.6. *Prove that* $\lim_{t \to \infty} \varphi(t, t_0, x_0) = 0$.

Remark 4.5. *Note that the condition of the function $V(t,x)$ to be decrescent can not be omitted in the second Lyapunov theorem. In order to show this let us consider the Massera's example. We construct a differential equation, which satisfies all conditions of the second Lyapunov theorem except the condition for the function $V(t,x)$ to be decrescent.*

Let us construct the following function $g(t)$:

(a) at integer $t = n$, $g^2(n) = 1, \ \forall n$,

(b) in the intervals $\left[(n-1) + \frac{1}{2}\left(\frac{1}{n-1}\right)^{(n-1)}, n - \frac{1}{2}\left(\frac{1}{n}\right)^n\right]$ the function

$$g^2(t) = e^{-t}, \ t \in \left[(n-1) + \frac{1}{2}\left(\frac{1}{n-1}\right)^{(n-1)}, n - \frac{1}{2}\left(\frac{1}{n}\right)^n\right],$$

(c) on the interval $(n - \frac{1}{2}(\frac{1}{n})^n, n + \frac{1}{2}(\frac{1}{n})^n)$ the function $g^2(t)$ is joined with the function e^{-t} and 1 by continuously differentiable way.

The function $g(t)$ is continuously differentiable on $(0, \infty)$.

For the equation

$$\dot{x} = \frac{g'(t)}{g(t)} x \tag{4.3}$$

the Lyapunov function

$$V(t,x) = \frac{x^2}{g^2(t)}(3 - \int_0^t g^2(s)ds \ )$$

satisfies all conditions of the second Lyapunov theorem except the condition for the function $V(t,x)$ to be decrescent. Note that $\dot{V}(t,x) = -x^2$. Because any solution of equation (4.3) has representation $x = c \ g(t), \ t \in (0, \infty)$, then the trivial solution is stable, but it is not asymptoticly stable.

4.2.2 The Chetaev theorem

Theorem 4.5. *(Chetaev). The equilibrium $x = 0$ of a system $\dot{x} = F(t,x)$ is unstable if there exists a function $V(t,x)$ with the properties:*

(a) $V \in C^1(D_h)$,

(b) V is bounded on the $\Pi = \{(t,x) \in D_h | \ V(t,x) > 0\}$,

(c) $\dot{V}(t,x) > 0$ in Π,

(d) $\forall \alpha > 0, \ \exists \beta(\alpha) > 0$, that $\dot{V}(t,x) \geq \beta$ on Π_α, where $\Pi_\alpha = \{(t,x) \in D_h | \ V(t,x) \geq \alpha\}$,

(e) $\exists t_0$ such that $0 \in \bar{D}_{t_0}$, where $D_{t_0} = \{x \in R^n | \ (t_0, x) \in \Pi\}$.

Proof.

In order to prove the theorem one needs to find $\hat{t}_0$ and $\varepsilon > 0$ such that

$$\forall\delta > 0\ \exists x_0,\ |x_0| < \delta,\ \exists t_1 > \hat{t}_0 \implies |\varphi(t_1)| \geq \varepsilon.$$

Let $\varepsilon = h$ (the radius of the cylinder D_h), $\hat{t}_0 = t_0$ (from property (e)). Because $0 \in \bar{D}_{t_0}$, then for any $\delta > 0\ \exists\ x_0 \in D_{t_0}$ and $|x_0| < \delta$. Let us consider the solution $x = \varphi(t, t_0, x_0)$ of the Cauchy problem (4.1). Note that

$$v(t_0) = V(t_0, x_0) > 0.$$

Assume that $|\varphi(t, t_0, x_0)| < \varepsilon,\ \forall t > t_0$. It will be proven that this contradicts to the conditions of the theorem.

Let the set

$$S = \{t > t_0 \mid v(t) = 0\} \neq \emptyset.$$

Denote t_* the nearest point to t_0: $t_* = \inf\ S$. Note that $v(t_*) = 0$ and $v(t) = V(t, \varphi(t, t_0, x_0)) > 0$ in the interval $[t_0, t_*)$. Thus, $(t, \varphi(t, t_0, x_0)) \in \Pi$ and $\dot{v}(t) = \dot{V}(t, \varphi(t, t_0, x_0)) > 0$ (according to the third property of the theorem). It implies $v(t) > v(t_0),\ \forall t > t_0$. For example, for $t = t_*$: $v(t_*) > v(t_0) > 0$, which contradicts to the condition $v(t_*) = 0$. Therefore, the set S is empty, $(t, \varphi(t, t_0, x_0)) \in \Pi$ and $v(t) > v(t_0),\ \forall t > t_0$.

Let $\alpha = v(t_0) > 0$. Note that

$$v(t) > v(t_0) = \alpha, \quad \forall t > t_0.$$

Since $\varphi(t, t_0, x_0)) \in D_h$ and $v(t) > 0$, one has $(t, \varphi(t)) \in \Pi_\alpha$. Using property (d) there is $\beta = \beta(\alpha) > 0$ that $\dot{V}(t, \varphi(t, t_0, x_0)) \geq \beta$ and then

$$v(t) = v(t_0) + \int_{t_0}^{t} \dot{v}(s)\ ds\ \geq v(t_0) + \beta(t - t_0).$$

This contradicts to the boundness of $V(t, x)$. □

Exercise 4.7. *Using the Chetaev theorem (taking $V(t, x) = x^2$) prove that the solution $x = 0$ of the equation*

$$\dot{x} = ax, \quad a > 0$$

is not stable.

4.3 Stability of quasilinear systems

Definition 4.8. *A system of ODE's $\dot{x} = F(t, x)$ is called a quasilinear system if*

$$F(t, x) = A(t)x + \varphi(t, x),$$

where $A(t)$ is $m \times m$ matrix and the function $\varphi(t, x)$ satisfies the conditions:

(a) $\varphi(t, 0) = 0$,

(b) $\displaystyle\lim_{|x|\to 0} \frac{|\varphi(t, x)|}{|x|} = 0$ uniformly with respect to t.

Let the matrix A be a constant matrix.

Theorem 4.6. *If all eigenvalues of the constant matrix A have strictly negative real parts, then the trivial solution of the quasilinear system $\dot{x} = F(t, x)$ is asymptoticly stable.*

Proof.

We prove the theorem by applying the second Lyapunov theorem.

Let us consider the linear homogeneous system

$$\dot{y} = Ay.$$

Assume that $\Phi(t)$ is a fundamental matrix of the system. By virtue of the construction of a fundamental matrix, every element of the fundamental matrix $\Phi(t)$ has the resprsentation

$$P(t) \cdot e^{\lambda t}$$

where $P(t)$ is a polynomial of the degree less than multiplicity of the eigenvalue λ of the matrix A. Because $Re(\lambda) < 0$, then the integral

$$\int_0^\infty \Phi^*(\tau)\Phi(\tau)\, d\tau$$

converges. Since fundamental matrices are related by $\hat{\Phi}(t) = \Phi(t)C$ with some constant matrix C, without loss of generality, we can account that $\Phi(t)$ is a matrizant: a fundamental matrix with the identical matrix $\Phi(0)$.

Choose

$$V(t,x) \equiv V(x) = \int_0^\infty (\Phi(\tau)x, \Phi(\tau)x)d\tau = \int_0^\infty (\Phi^*(\tau)\Phi(\tau)x, x)d\tau = \left(\left(\int_0^\infty \Phi^*(\tau)\Phi(\tau)d\tau\right)x, x\right)$$

or

$$V(t,x) = (Sx, x),$$

where

$$S = \int_0^\infty \Phi^*(\tau)\Phi(\tau)d\tau$$

is a numerical matrix. Since the matrix $\Phi(t)$ is nonsingular and continuous the scalar product $(Sx, x) \geq 0$, and $(Sx, x) = 0$ if only if $x = 0$.

The matrix S is equivalent to the diagonal matrix Λ with positive diagonal elements λ_i ($i = 1, 2, \ldots, m$). Even more, because S is a symmetric matrix, there is an orthogonal matrix P and a diagonal matrix Λ such that

$$S = P^*\Lambda P.$$

Let us consider a scalar product:

$$(Sx, x) = (P^*\Lambda Px, x) = (\Lambda Px, Px) = (\Lambda y, y) = \sum_{\alpha=1}^m \lambda_\alpha y_\alpha^2 \geq 0,$$

where the vector $y = Px$. Because the equality $(Sx, x) = 0$ is only possible if $x = 0$, then $\lambda_i > 0$. Even more

$$\left(\max_i(\lambda_i)\right) x^2 \geq (Sx, x) \geq \left(\min_i(\lambda_i)\right) x^2 > 0, \quad \forall x \neq 0.$$

This means that the function $V(t,x)$ is a decrescent, positive definite function. Note also that $\nabla V(x) = 2Sx$.

The solution $\xi(t,x) = \Phi(t)x$ of the Cauchy problem

$$\begin{cases} \dot{y} = Ay, \\ y(0) = x \end{cases}$$

has the property

$$\xi(\tau, \xi(t,x)) = \xi(t + \tau, x).$$

In fact, let $a(\tau) = \xi(\tau, \xi(t,x))$ and $b(\tau) = \xi(t+\tau, x)$. Then

$$a(0) = \xi(0, \xi(t,x)) = \xi(t,x), \quad b(0) = \xi(t,x)$$

and

$$\frac{d}{d\tau}a = Aa, \quad \frac{d}{d\tau}b = Ab.$$

By virtue of the uniqueness of a solution of the Cauchy problem one obtains $a(\tau) = b(\tau)$.

In order to use the second Lyapunov theorem one needs to show that the function

$$\dot V(t,x) = (\nabla V(x), Ax + \varphi(t,x)) = (\nabla V(x), Ax) + (\nabla V(x), \varphi(t,x)).$$

is negative definite.

For the first term one has

$$(\nabla V(x), Ax) = \left(\nabla V(\xi(t,x)), \frac{d\xi}{dt}\right)_{t=0} = \left(\frac{d}{dt}(V(\xi(t,x)))\right)_{t=0}.$$

But

$$V(\xi(t,x)) = \int_0^\infty (\Phi(\tau)\xi(t,x))^2 d\tau = \int_0^\infty (\xi(t+\tau,x))^2 d\tau = \int_t^\infty (\xi(\tau,x))^2 d\tau,$$

therefore

$$\frac{d}{dt}(V(\xi(t,x))) = -(\xi(t,x))^2$$

or

$$(\nabla V(x), Ax) = -x^2.$$

The second term $(\nabla V(x), \varphi(t,x))$ satisfies the inequality:

$$|(\nabla V(x), \varphi(t,x))| \le |\nabla V||\varphi(t,x)| \le 2 \parallel S \parallel |x| \; |\varphi(t,x)|.$$

Because $\lim\limits_{|x|\to 0} \dfrac{|\varphi(t,x)|}{|x|} = 0$ uniformly with respect to t, then

$$\forall \varepsilon > 0 \; \exists \; \delta > 0, \; \forall x, |x| < \delta \Longrightarrow |\varphi(t,x)| < \varepsilon |x|.$$

If ε satisfies the inequality

$$\alpha = 1 - 2 \parallel S \parallel \varepsilon > 0,$$

then in the cylinder

$$D_\delta = \{(t,x)| \; t > a, \quad |x| \le \delta\}$$

the function $\dot V(t,x)$ is negative definite:

$$\dot V(t,x) = -x^2 + 2(Sx, \varphi(t,x)) < -x^2 + (2 \parallel S \parallel \varepsilon)x^2 < -\alpha x^2.$$

By virtue of the second Lyapunov theorem the trivial solution is asymptotically stable. □

Corollary 4.1. *If all eigenvalues of the Jacobi matrix*

$$A = \frac{\partial F(x)}{\partial x}_{|x=0}$$

are strictly negative, then the trivial solution of an autonomous system

$$\dot x = F(x), \quad F \in C^2$$

is asymptoticly stable.

Proof.

Because $F \in C^2$, then from the Taylor formula

$$F(x) = Ax + \varphi(x)$$

with some function $\varphi(x) \in C^1$, which satisfies

$$\lim_{x \to 0} \frac{|\varphi(x)|}{|x|} = 0.$$

The proof of the corollary follows from the previous theorem. $\square$

Theorem 4.7. *The trivial solution of a linear system*

$$\dot{x} = Ax$$

with a constant matrix A is unstable if at least one of eigenvalues has strictly positive real part.

Exercise 4.8. *Prove this theorem.*

Hint: find a solution, which corresponds to an eigenvalue with a positive real part.

Definition 4.9. *A critical point x_0 $(F(t, x_0) = 0)$ is called totally unstable, if $\forall t_0$ there exists $\delta > 0$ such that*

$$\forall \xi, \ |\xi - x_0| < \delta, \ \exists \, T = T(t_0, \xi) \ \text{that} \ \forall t > T \Longrightarrow |\varphi(t, t_0, \xi) - x_0| \geq \delta$$

where $x = \varphi(t, t_0, \xi)$ is a solution of the Cauchy problem

$$\dot{x} = F(t, x), \quad x(t_0) = \xi.$$

Theorem 4.8. *Let a quasilinear system*

$$\dot{x} = F(t, x) = Ax + \varphi(t, x)$$

has a constant matrix A. If all eigenvalues of the matrix A have a strictly positive real part, then the trivial solution $x = 0$ of this system is totally unstable.

Proof.

Take the system

$$\dot{x} = -Ax.$$

With the help of a fundamental matrix of this system one can construct the Lyapunov function

$$V(x) = (Sx, x).$$

This function has the properties (the proof is the same as in the previous theorem):

(a) $(\nabla V, -Ax) = (2Sx, -Ax) = -x^2$,

(b) $\alpha_2 x^2 \leq V(x) \leq \alpha_1 x^2$, $\quad \alpha_1 > 0, \ \alpha_2 > 0$, where $\alpha_1 = \max(\lambda_k), \ \alpha_2 = \min(\lambda_k)$.

For the function $\dot{V}(x)$ one has

$$\dot{V}(x) = (\nabla V, Ax + \varphi(t, x)) = (\nabla V, Ax) + (\nabla V, \varphi(t, x)) = x^2 + 2(Sx, \varphi(t, x)).$$

Because the system $\dot{x} = F(t, x)$ is a quasilinear system, then

$$\forall \varepsilon > 0, \ \exists \, \delta_1 > 0, \ \forall x, \ |x| < \delta_1 \Longrightarrow |\varphi(x, t)| < \varepsilon |x|.$$

Therefore, if $|x| < \delta_1$, then

$$|(\nabla V, \varphi(t,x))| \le 2 \parallel S \parallel \varepsilon x^2$$

or

$$-(2 \parallel S \parallel \varepsilon)x^2 \le (\nabla V, \varphi(t,x)) \le (2\varepsilon \parallel S \parallel)x^2.$$

Suppose that ε satisfies the inequality

$$\alpha = 1 - 2\varepsilon \parallel S \parallel > 0,$$

then

$$\dot{V}(x) \ge \alpha x^2 = \frac{\alpha}{\alpha_1}\alpha_1 x^2 \ge \beta V(x),$$

where $\beta = \alpha/\alpha_1 > 0$.

There exists a number $c \le \varepsilon$, such that $\forall x$ satisfying the inequality $V(x) \le c$, then $|x| \le \delta_1$. In fact, assume opposite, that

$$\forall c \le \varepsilon,\ \exists x,\ \text{ such that if } V(x) \le c \implies |x| > \delta_1.$$

Let $c_n = 1/n$, then there is a sequence $\{x_n\}$ that

$$V(x_n) \le \frac{1}{n}, \qquad |x_n| > \delta_1.$$

But $V(x) \ge \alpha_2 x^2$, therefore $|x_n| \underset{n\to\infty}{\to} 0$. It contradicts to the condition that $|x_n| > \delta_1$. Thus, there is the number $c \le \varepsilon$ that for any x satisfying the inequality $V(x) \le c$ one has

$$\dot{V}(x) \ge \beta V(x)$$

Let

$$\delta = \min(\delta_1, \sqrt{\frac{c}{\alpha_1}})$$

and the cylinder

$$D = \{(t,x) | t > a,\ |x \le \delta\}.$$

Note that if $V(x) \ge c$, then $|x| \ge \sqrt{c/\alpha_1} \ge \delta$.

Assume that $|\xi| < \delta$, $x = h(t,\xi)$ is a solution of the Cauchy problem

$$\begin{cases} \dot{x} = F(t,x), \\ x(t_0) = \xi \end{cases}$$

and $v(t) = V(h(t,\xi))$. Because $\xi \ne 0$, then $v(t_0) > 0$ and

$$v(t) \ge v(t_0)e^{\beta(t-t_0)}$$

or

$$V(h(t,\xi)) \ge V(\xi)e^{\beta(t-t_0)}$$

From the last inequality one can conclude that there is $T = T(t_0, \xi)$ such that

$$V(h(T,\xi)) = c.$$

Now we will prove that if $t > T$ and the solution $x = h(t,\xi)$ is definite for these values of t, then $V(h(t,\xi)) \ge c$. Assume opposite:

$$\exists t_1 > T \text{ that } V(h(t_1,\xi)) < c.$$

Let $t_2 < t_1$ is the nearest point to t_1, where $V(h(t_2, \xi)) = c$ and

$$V(h(t, \xi)) \le c \quad \forall t \in [t_2, t_1].$$

There is $t_* \in [t_2, t_1]$ such that

$$\dot{v}(t_*) = \frac{v(t_1) - v(t_2)}{t_1 - t_2} \le 0.$$

Since $t_* \in [t_1, t_2]$, then $0 < V(h(t_*, \xi)) \le c$, and therefore

$$\dot{v}(t_*) = \dot{V}(h(t_*, \xi)) \ge \beta V(h(t_*, \xi)) > 0.$$

One obtains a contradiction.

Thus, $\forall t > T$ the solution $x = h(t, \xi)$ satisfies the inequality $V(h(t, \xi)) \ge c$. This gives $|h(t, \xi)| \ge \delta$, $\forall t > T$. $\square$

Theorem 4.9. *If at least one of eigenvalues of a constant matrix A has strictly positive real part, then the trivial solution of the quasilinear system*

$$\dot{x} = Ax + \varphi(t, x)$$

is unstable.

Exercise 4.9. *Prove this theorem.*
Hint: use the Chetaev theorem.

Chapter 5

Introduction to functional differential equations

This chapter gives an introduction to delay differential equations[1].

5.1 Definitions

A more general type of differential equations than considered up to now is type of equations which is called a functional differential equations. The simplest type of functional differential equations is "delay differential system of equations" such as

$$\dot{x}(t) = G(t, x(g_1(t)), x(g_2(t)), \ldots, x(g_n(t))), \tag{5.1}$$

where $x \in R^m,\ g_j(t) \in [t-r, t],\quad \forall t \geq t_0,\quad (j = 1, 2, \ldots, n)$, for some constant $r \geq 0$. For example, $g_1(t) = t - 1,\ g_2(t) = t - 2,\ n = 2,\ m = 1$ and the equation is

$$\dot{x}(t) = 3x(t-1) - x(t-2).$$

Definition 5.1. *If $\chi(t)$ is a function defined at least on $[t-r, t]$, then a new function $\chi_t : [-r, 0] \to R^m$ is defined by*

$$\chi_t(s) = \chi(t+s), \quad s \in [-r, 0].$$

We will denote the set of continuous on $[-r, 0]$ functions with values in D by

$$\mathcal{Q}_D = \{\chi \in C([-r, 0]) \mid \chi(t) \in D \subset R^m,\ \forall t \in [-r, 0]\}.$$

Definition 5.2. *The equation*

$$\dot{x}(t) = F(t, x_t) \tag{5.2}$$

with the functional $F : J \times \mathcal{Q}_D \to R^m$ is called a functional differential equation. Here J is some interval (α, β).

For any function $\chi \in \mathcal{Q}_D$ we define the value

$$\| \chi \|_r = \sup_{-r \leq s \leq 0} |\chi(s)|,$$

[1] We follow to the textbook: R.D.Driver. Ordinary and delay differential equations. Springer–Verlag New York Inc., 1977.

which can be considered as a norm of the Banach space $\mathcal{Q}_{R^m}$ containing the space of functions $\chi \in \mathcal{Q}_D$.

Example 5.1. For system of equations (5.1) the functional $F(t, x_t)$ is defined as

$$F(t, x_t) = G(t, x_t(g_1(t) - t), x_t(g_2(t) - t), \ldots, x_t(g_n(t) - t)).$$

Example 5.2. If the functional is $F(t, \chi) = \int_{-r}^{0} \chi(s)\, ds$, then the functional differential equation is

$$\dot{x}(t) = \int_{-r}^{0} x_t(s)\, ds = \int_{t-r}^{t} x(s)\, ds.$$

When we were studying existence and uniqueness for system of ordinary differential equations we had needed to consider a continuity and Lipschitz conditions. These are replaced for a functional $F(t, x_t)$ by the following way.

Definition 5.3. *A continuity condition is satisfied if the function $F(t, \chi_t)$ is continuous function with respect to $t \in [t_0, \beta)$ for each given continuous function $\chi : [t_0 - r, \beta) \to D$.*

Let $\mathcal{Q}$ be a subset of $J \times \mathcal{Q}_D$.

Definition 5.4. *The functional $F : J \times \mathcal{Q}_D \to R^m$ satisfies a Lipschitz condition on $\mathcal{Q}$ (or F is Lipschitzian on $\mathcal{Q}$) if there exists a positive constant L that*

$$|F(t, \chi) - F(t, \hat{\chi})| \le L \parallel \chi - \hat{\chi} \parallel_r,$$

for arbitrary (t, χ) and $(t, \hat{\chi})$ from $\mathcal{Q}$.

Definition 5.5. *The functional $F : J \times \mathcal{Q}_D \to R^m$ is said to be a locally Lipschitzian if for each given $(t_0, \chi_0) \in J \times \mathcal{Q}_D$ there exist numbers $a > 0$ and $b > 0$ such that F is Lipschitzian on the subset*

$$\mathcal{Q} = \{(t, \chi) \in J \times \mathcal{Q}_D \mid \ t \in [t_0 - a, t_0 + a] \cap J, \ \ \chi \in \mathcal{Q}_D, \ \ \parallel \chi - \chi_0 \parallel_r \le b\}.$$

Exercise 5.1. *Prove that if the function $F : J \times D^n \to R^m$ satisfies the Lipschitz condition (as a function), then it satisfies the Lipschitz condition as the functional $F : J \times \mathcal{Q}_D \to R^m$.*

Definition 5.6. *The problem of finding a solution of system (5.2), which satisfies the values*

$$x_{t_0}(s) = \psi(s), \quad s \in [t_0 - r, t_0] \tag{5.3}$$

is called an initial value problem.

5.2 Uniqueness of solution

The uniqueness theorem will use the following result.

Lemma 5.1. *Let $\chi : [t_0 - r, \beta) \to R^m$ be continuous. Then for any $\hat{t} \in [t_0, \beta)$ and any $\varepsilon > 0$ there exists $\delta = \delta(\hat{t}, \varepsilon) > 0$ such that $\forall t \in [t_0, \beta)$ and $|t - \hat{t}| < \delta$:*

$$\parallel \chi_t - \chi_{\hat{t}} \parallel_r < \varepsilon.$$

Proof.

Let $\hat{t} \in [t_0, \beta)$ and ε are arbitrary. Choose β_1 that $\hat{t} < \beta_1 < \beta$. Since χ is uniformly continuous on the closed bounded interval $[t_0 - r, \beta_1]$, there exists $\delta \in (0, \beta_1 - \hat{t})$ such that if τ and $\hat{\tau}$ belong to the interval $[t_0 - r, \beta_1]$ and $|\tau - \hat{\tau}| < \delta$, then

$$|\chi(\tau) - \chi(\hat{\tau})| < \varepsilon/2.$$

If $t \in [t_0 - r, \beta)$ such that $|t - \hat{t}| < \delta$, then $t \in [t_0 - r, \beta_1)$ and, since $|t + s - (\hat{t} + s)| = |t - \hat{t}| < \delta$, we have

$$|\chi_t(s) - \chi_{\hat{t}}(s)| = |\chi(t + s) - \chi(\hat{t} + s)| < \varepsilon/2, \quad \forall s \in [-r, 0].$$

Therefore

$$\| \chi_t - \chi_{\hat{t}} \|_r = \sup_{-r \le s \le 0} |\chi_t(s) - \chi_{\hat{t}}(s)| \le \varepsilon/2 < \varepsilon.$$

□

Note that if the functional $F : [t_0, \beta) \times \mathcal{Q}_D \to R^m$ satisfies continuity condition with respect to t in $[t_0, \beta)$, then the continuous function $\varphi : [t_0 - r, \beta) \to D$ is a solution of the initial value problem (5.2), (5.3) if and only if

$$\varphi(t) = \begin{cases} \psi(t - t_0), & \forall t \in [t_0 - r, t_0], \\ \psi(0) + \int_{t_0}^{t} F(s, \varphi_s)\, ds, & \forall t \in [t_0, \beta). \end{cases}$$

Theorem 5.1. *If the functional $F : [t_0, \beta) \times \mathcal{Q}_D \to R^m$ satisfies the continuity condition and it is locally Lipschitzian. Then the initial value problem (5.2), (5.3) with $\psi \in \mathcal{Q}_D$ has at most one solution on $[t_0 - r, \beta_1)$, for any $\beta_1 \in (t_0, \beta]$.*

Proof.

Suppose that for some $\beta_1 \in (t_0, \beta]$ there are two different solutions $x = \varphi(t)$ and $x = \hat{\varphi}(t)$:

$$\varphi : [t_0 - r, \beta_1) \to D, \quad \hat{\varphi} : [t_0 - r, \beta_1) \to D.$$

Denote

$$t_1 = \inf\{t \in [t_0, \beta_1) \mid \varphi(t) \ne \hat{\varphi}(t)\}.$$

Note that $t_0 \le t_1 < \beta_1$ and

$$\varphi(t) = \hat{\varphi}(t), \quad \forall t \in [t_0 - r, t_1]$$

or $\varphi_{t_1} = \hat{\varphi}_{t_1}$. Since $(t_1, \varphi_{t_1}) \in [t_0, \beta_1) \times \mathcal{Q}_D$, there exist numbers $a > 0$ and $b > 0$ such that the set

$$\mathcal{Q} = \{(t, \chi) \in J \times \mathcal{Q}_D \mid \ t \in [t_1, t_1 + a], \ \ \chi \in \mathcal{Q}_D, \ \ \| \chi - \varphi_{t_1} \|_r \le b\}$$

is a subset of $[t_0, \beta) \times \mathcal{Q}_D$ and F is Lipschitzian on $\mathcal{Q}$ (with the Lipschitz constant L).

By the previous lemma, there exists $\delta > 0$ ($\delta < a$) such that $(t, \varphi_t) \in \mathcal{Q}$ and $(t, \hat{\varphi}_t) \in \mathcal{Q}$ for all $t \in [t_1, t_1 + \delta)$. Since $\varphi(t)$ and $\hat{\varphi}(t)$, $\forall t \in [t_0, \beta_1)$ are solutions of system (5.2), we have

$$|\varphi(t) - \hat{\varphi}(t)| \le |\varphi(t_0) - \hat{\varphi}(t_0)| + \int_{t_0}^{t} |\varphi'(s) - \hat{\varphi}'(s)|\, ds = \int_{t_1}^{t} |\varphi'(s) - \hat{\varphi}'(s)|\, ds =$$

$$\int_{t_1}^{t} |F(s, \varphi_s) - F(s, \hat{\varphi}_s)|\, ds.$$

If $t \in [t_1, t_1 + \delta)$, then

$$|\varphi(t) - \hat{\varphi}(t)| \le \int_{t_1}^{t} L \| \varphi_s - \hat{\varphi}_s \|_r \ ds.$$

Therefore, if $\forall t \in [t_1, t_1 + \delta)$, then

$$\| \varphi_t - \hat{\varphi}_t \|_r = \sup_{\sigma \in [-r,0]} |\varphi(t+\sigma) - \hat{\varphi}(t+\sigma)| = \sup_{\sigma \in [-r,0],\ \sigma > t_1 - t} |\varphi(t+\sigma) - \hat{\varphi}(t+\sigma)| \le \int_{t_1}^{t} L \| \varphi_s - \hat{\varphi}_s \|_r \ ds.$$

From this it follows that $\| \varphi_t - \hat{\varphi}_t \|_r = 0$ for any $t \in [t_1, t_1 + \delta)$ or $\varphi(t) = \hat{\varphi}(t)$ for any $t \in [t_1, t_1 + \delta)$. This contradicts to the definition of t_1. □

5.3 Existence theorem

Here we denote $J = [t_0 - r, \beta)$.

Theorem 5.2. *Let the functional $F : [t_0, \beta) \times \mathcal{Q}_D \to R^m$ satisfy continuity condition with respect to t in $[t_0, \beta)$ and let it be locally Lipschitzian. Then the initial value problem (5.2), (5.3) with $\psi \in \mathcal{Q}_D$ has an unique solution on $[t_0 - r, t_0 + \Delta)$, for some $\Delta > 0$.*

Proof.

Since $(t_0, \psi) \in J \times \mathcal{Q}_D$ and the functional $F : [t_0, \beta) \times \mathcal{Q}_D \to R^m$ is a locally Lipschitzian (with the constant L), there are constants $a > 0$ and $b > 0$ that F is Lipschitzian on the subset

$$\mathcal{Q} = \{(t, \chi) \in J \times \mathcal{Q}_D \mid \ t \in [t_0 - a, t_0 + a] \cap J, \ \ \chi \in \mathcal{Q}_D, \ \ \| \chi - \psi \|_r \leq b\}.$$

Define a continuous function $\psi^0(t)$ on $J = [t_0 - r, \beta]$ by

$$\psi^0(t) = \begin{cases} \psi(t - t_0), & \forall t \in [t_0 - r, t_0], \\ \psi(0), & \forall t \in [t_0, \beta). \end{cases}$$

The function $F(t, \psi^0_t)$ continuously depends on t, hence there exists some constant B_1 that

$$|F(t, \psi^0_t)| \leq B_1, \quad \forall t \in [t_0, t_0 + a].$$

Let $B = Lb + B_1$. There exists a constant $a_1 \in (0, a]$ such that $\forall t \in [t_0, t_0 + a_1]$:

$$\| \psi^0_t - \psi \|_r = \| \psi^0_t - \psi^0_{t_0} \|_r \leq b.$$

Choose $\Delta > 0$ such that

$$\Delta \leq \min(a_1, b/B).$$

Remained part of the proof is the same as the proof of the Picard theorem.

Let U be the set of functions $\{\chi(t) \in C(J_1)\}$ with the norm

$$\| \chi \| = \max_{t \in J_1} (e^{-L|t - t_0|} |\chi(t)|),$$

where $J_1 = [t_0 - r, t_0 + \Delta]$. This norm is equivalent to the uniform norm on the space of continuous functions $C(J_1)$:

$$\| \chi \|_1 = \max_{t \in J_1} |\chi(t)|.$$

Therefore $(U, \| \cdot \|)$ is a Banach space.

Define the closed set M by

$$M = \{\chi(t) \in U \mid \ \chi(t) = \psi(t - t_0), \ \forall t \in [t_0 - r, t_0], \ \ |\chi(t) - \psi(0)| \leq b, \ \forall t \in [t_0, t_0 + \Delta]\}$$

Note that if $\chi \in M$ and $t \in [t_0, t_0 + \Delta]$, then $\| \chi_t - \psi^0_t \|_r \leq b$. Therefore $(t, \chi_t) \in \mathcal{Q}$ and

$$|F(t, \chi_t)| \leq |F(t, \chi_t) - F(t, \psi^0_t)| + |F(t, \psi^0_t)| \leq L \| \chi_t - \psi^0_t \|_r + B_1 \leq B.$$

For each $\chi \in M$ define a mapping $T\chi$ by

$$T\chi(t) = \begin{cases} \psi(t - t_0), & \forall t \in [t_0 - r, t_0], \\ \psi(0) + \int_{t_0}^{t} F(s, \chi_s) \, ds, & \forall t \in [t_0, t_0 + \Delta]. \end{cases}$$

Since $|F(s, \chi_s)| \leq B$, there is

$$|T\chi(t) - \psi(0)| \leq B\Delta \leq b, \quad \forall t \in [t_0, t_0 + \Delta].$$

Also $T\chi(t)$ is continuous. Thus $T\chi(t) \in M$ and we can conclude that $T : M \to M$.

For proving the theorem it is remained to prove that $T : M \to M$ is a contraction. In order to do it we need to find a constant q that $0 < q < 1$ and $\forall \chi^1(t) \in M, \chi^2(t) \in M$ we have:

$$\| T\chi^1 - T\chi^2 \| < q \| \chi^1 - \chi^2 \| .$$

Since $(t, \chi^1_t) \in \mathcal{Q}$, $(t, \chi^2_t) \in \mathcal{Q}$ for all $t \in [t_0, t_0 + \Delta)$, we have that

$$e^{-L(t-t_0)}|T\chi^1(t) - T\chi^2(t)| \leq e^{-L(t-t_0)} \int_{t_0}^{t} |F(s, \chi^1_s) - F(s, \chi^2_s)|\, ds \leq Le^{-L(t-t_0)} \int_{t_0}^{t} \| \chi^1_s - \chi^2_s \|_r \, ds =$$

$$= Le^{-L(t-t_0)} \int_{t_0}^{t} \max_{\sigma\in[-r,0]} e^{-L(s-t_0+\sigma)} e^{L(s-t_0+\sigma)} |\chi^1(s+\sigma) - \chi^2(s\sigma)|\, ds \leq$$

$$\leq Le^{-L(t-t_0)} \int_{t_0}^{t} e^{L(s-t_0)} \max_{\sigma\in[-r,0]} e^{-L(s-t_0+\sigma)} |\chi^1(s+\sigma) - \chi^2(s+\sigma)|\, ds \leq$$

$$\leq Le^{-L(t-t_0)} \int_{t_0}^{t} e^{L(s-t_0)} \| \chi^1 - \chi^2 \| \, ds = Le^{-L(t-t_0)} \| \chi^1 - \chi^2 \| \int_{t_0}^{t} e^{L(s-t_0)}\, ds =$$

$$= (1 - e^{-L(t-t_0)}) \| \chi^1 - \chi^2 \| \leq (1 - e^{-L\Delta}) \| \chi^1 - \chi^2 \|, \quad \forall t \in [t_0, t_0 + \Delta).$$

Because $T\chi^1(t) = T\chi^2(t)$ for $t \in [t_0 - r, t_0]$, then we obtained that

$$\| T\chi^1 - T\chi^2 \| \leq q \| \chi^1 - \chi^2 \|$$

with the constant $0 < q = 1 - e^{-L\Delta} < 1$. Thus, we have constructed a contraction operator $T : M \to M$ with the closed set $M \subset U$, where U is a Banach space.

From the contraction principle we can conclude that there exists $\chi(t) \in M$ such that

$$\chi(t) = \begin{cases} \psi(t - t_0), & \forall t \in [t_0 - r, t_0], \\ \psi(0) + \int_{t_0}^{t} F(s, \chi_s)\, ds, & \forall t \in [t_0, t_0 + \Delta]. \end{cases}$$

□

The following definitions and theorems are like their counterparts for ordinary differential equations.

Definition 5.7. *Let $\phi(t)$ on $[t_0 - r, \beta_1)$ and $\hat{\phi}(t)$ on $[t_0 - r, \beta_2)$ both be solutions of problem (5.2), (5.3). If $\beta_2 > \beta_1$, then the solution $\hat{\phi}(t)$ is called an extension of $\phi(t)$ to $[t_0 - r, \beta_2)$. A solution $\phi(t)$ of (5.2), (5.3) is nonextendable if it has no extension.*

Theorem 5.3. *Let $F : [t_0, \beta) \times \mathcal{Q}_D \to R^m$ satisfy the continuity condition, and let it be locally Lipschitzian. Then for each $\psi \in \mathcal{Q}_D$, problem (5.2), (5.3) has an unique nonextendable solution.*

Exercise 5.2. *Prove the theorem.*

Lemma 5.2. *Let $\phi(t)$ be a differentiable function on the bounded open interval (α, β) with*

$$|\phi'(t)| \leq B, \quad t \in (\alpha, \beta).$$

Then there exist $\lim_{t\to\alpha} \phi(t)$ and $\lim_{t\to\beta} \phi(t)$.

Exercise 5.3. *Prove the lemma.*

The theorem that describes a behavior of nonextendable solution $\varphi(t)$, $t \in [t_0 - r, \beta_1)$ at the end $t = \beta_1$ requires additional hypothesis.

Definition 5.8. *The functional $F : [t_0, \beta) \times \mathcal{Q}_D \to R^m$ is said to be a quasi–bounded if F is bounded on every set of the form $[t_0, \beta_1) \times \mathcal{Q}_A$ where $t_0 < \beta_1 < \beta$ and A is a closed bounded subset of D.*

Because for delay equations we have not proven theorem of continuity with respect to initial values, we can prove less than it was proven for ordinary differential equations.

Theorem 5.4. *(behavior of nonextendable solution at the end). If $F : [t_0, \beta) \times \mathcal{Q}_D \to R^m$ satisfies the continuity condition, is locally Lipschitzian and quasi–bounded. Then for each closed bounded set $A \subset D$ and any nonextendable solution $\varphi(t)$, $t \in [t_0 - r, \beta_1)$ with $\beta_1 < \beta$ of problem (5.2), (5.3) there exists $t \in (t_0, \beta_1)$ that $\varphi(t) \notin A$.*

Proof.

Assume the opposite assertion that $\varphi(t) \in A, \quad \forall t \in [t_0, \beta_1)$.

Because F is a quasi–bounded and $\varphi(t) \in A \cup A_\psi \; \forall t \in [t_0 - r, \beta_1)$, then there is a constat B that

$$|F(t, \varphi_t)| \leq B, \quad \forall t \in [t_0, \beta_1).$$

This gives that

$$|\varphi'(t)| \leq B, \quad \forall t \in [t_0, \beta_1).$$

Hence, from the lemma we get that there exists

$$\lim_{t \to \beta_1} \varphi(t) = \varphi_*.$$

Note that $\varphi_* \in A$.

Let us extend the definition of the $\varphi(t)$ to a continuous function on the closed interval $[t_0 - r, \beta_1]$ by setting $\varphi(\beta_1) = \varphi_*$. Then from the continuity condition it follows that $F(t, \varphi_t)$ is continuous function on the interval $[t_0 - r, \beta_1]$. Thus the equation (5.2) extends to include the point $t = \beta_1$, i.e.,

$$\varphi(t) = \begin{cases} \psi(t - t_0), & \forall t \in [t_0 - r, t_0], \\ \psi(0) + \int_{t_0}^{t} F(s, \varphi_s)\, ds, & \forall t \in [t_0, \beta_1]. \end{cases}$$

Applying the existence theorem to the initial value problem

$$\begin{cases} \dot{z} = F(t, z_t) \\ z_{\beta_1} = \varphi_{\beta_1} \end{cases} \tag{5.4}$$

we conclude that this new problem has a solution $z(t)$ on the interval $[\beta_1 - r, \beta_1 + \Delta)$ for some $\Delta > 0$. Thus

$$z(t) = \begin{cases} \varphi(t), & \forall t \in [\beta_1 - r, \beta_1], \\ \varphi(\beta_1) + \int_{\beta_1}^{t} F(s, z_s)\, ds, & \forall t \in [\beta_1, \beta_1 + \Delta). \end{cases}$$

If we define $z(t)$ for $t \in [t_0 - r, \beta_1 - r]$ as $z(t) = \varphi(t)$, then

$$z(t) = \begin{cases} \psi(t - t_0), & \forall t \in [t_0 - r, t_0], \\ \psi(0) + \int_{t_0}^{t} F(s, \varphi_s)\, ds, & \forall t \in [t_0, \beta_1), \\ \varphi(t_0) + \int_{t_0}^{\beta_1} F(s, z_s)\, ds + \int_{\beta_1}^{t} F(s, z_s)\, ds, & \forall t \in [\beta_1, \beta_1 + \Delta). \end{cases}$$

$$= \begin{cases} \psi(t - t_0), & \forall t \in [t_0 - r, t_0], \\ \psi(0) + \int_{t_0}^{t} F(s, \varphi_s)\, ds, & \forall t \in [t_0, \beta_1 + \Delta). \end{cases}$$

This gives that $z(t),\ t \in [t_0, \beta_1 + \Delta)$ is an extension of the solution $\varphi(t),\ t \in [t_0, \beta_1)$. It contradicts that the solution $\varphi(t),\ t \in [t_0, \beta_1)$ is nonextendable. $\square$